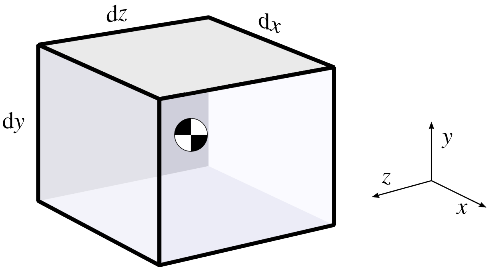
Balance of linear momentum

May the Force be

equal to the Mass times Acceleration

The Cauchy equation



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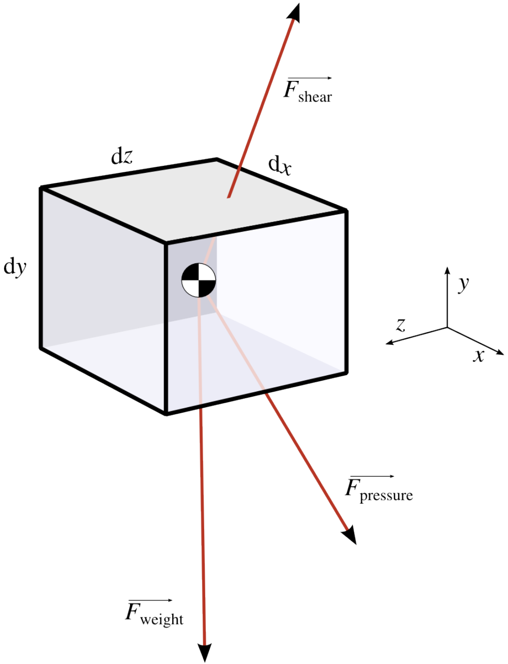


$\overrightarrow{F}_{\text{shear}}$
dz
dx
dy
y
z
x
$\overrightarrow{F}_{\text{pressure}}$
$\overrightarrow{F}_{\text{weight}}$

Newton from two points of view:

$$m_{\text{part.}}\frac{\mathrm{d}\vec{V}}{\mathrm{d}t} = \vec{F}_{\text{weight}} + \vec{F}_{\text{net, pressure}} + \vec{F}_{\text{net, shear}}$$

$$\rho\frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \rho\vec{g} + \frac{1}{\mathrm{d}\mathcal{V}}\vec{F}_{\text{net, pressure}} + \frac{1}{\mathrm{d}\mathcal{V}}\vec{F}_{\text{net, shear}}$$

$\longrightarrow$ How can we re-express pressure and shear?

Pressure force:

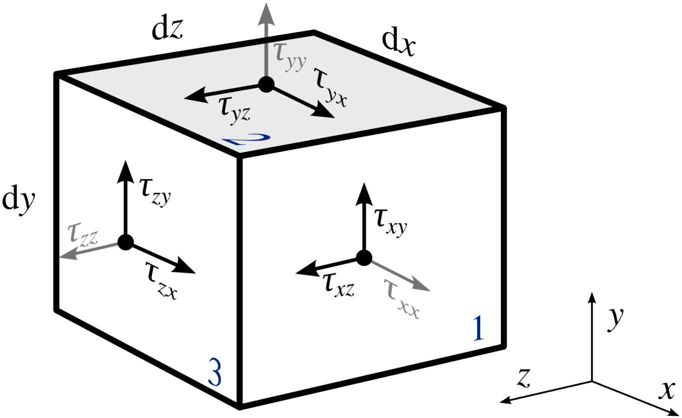
Back from previous chapters:

$$\frac{1}{\mathrm{d}\mathcal{V}}\vec{F}_{\text{net, pressure}} = -\vec{\nabla} p$$

Eulerian view of Newton

$$\rho \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \rho \vec{g} + \frac{1}{\mathrm{d}\mathcal{V}} \vec{F}_{\text{net, pressure}} + \frac{1}{\mathrm{d}\mathcal{V}} \vec{F}_{\text{net, shear}}$$

$\longrightarrow$ OK for pressure, what about shear?



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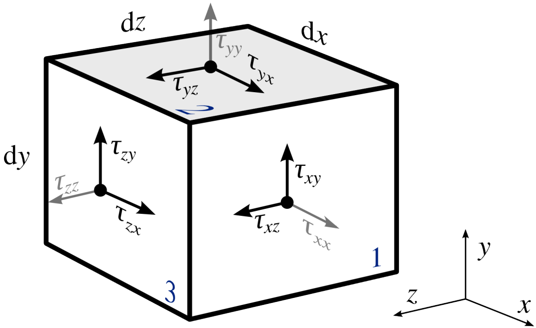
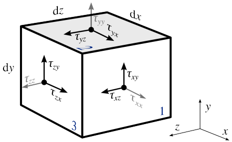
Shear in the x-direction:



$$\begin{aligned}\vec{F}_{\text{shear } x} &= \mathrm{d}\mathcal{V}\left(\frac{\partial\vec{\tau}_{zx}}{\partial z} + \frac{\partial\vec{\tau}_{yx}}{\partial y} + \frac{\partial\vec{\tau}_{xx}}{\partial x}\right) \\ &= \mathrm{d}\mathcal{V}\ \vec{\nabla}\cdot\vec{\tau}_{ix}\end{aligned}$$

Good memories from previous chapters



$$\vec{F}_{\text{shear}} = \mathrm{d}\mathcal{V} \begin{pmatrix} \vec{\nabla} \cdot \vec{\tau}_{ix} \\ \vec{\nabla} \cdot \vec{\tau}_{iy} \\ \vec{\nabla} \cdot \vec{\tau}_{iz} \end{pmatrix}$$

$$= \mathrm{d}\mathcal{V}\, \vec{\nabla} \cdot \vec{\tau}_{ij}$$

Aha!

We went from

$$\rho \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \rho \vec{g} + \frac{1}{\mathcal{V}} \vec{F}_{\text{net, pressure}} + \frac{1}{\mathcal{V}} \vec{F}_{\text{net, shear}}$$

to

$$\rho \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \rho \vec{g} - \vec{\nabla} p + \vec{\nabla} \cdot \vec{\tau}_{ij}$$

This is the *Cauchy equation*.

the Cauchy equation

$$\rho \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \rho\vec{g} - \vec{\nabla}p + \vec{\nabla}\cdot\vec{\tau}_{ij}$$

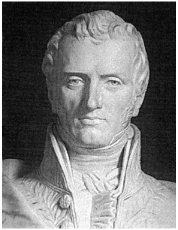
Regulates *all fluid flows*, all the time!

Now…

Can we eliminate $\vec{\nabla} \cdot \vec{\tau}_{ij}$ from this equation?

(how can we express the divergent of the shear tensor $\vec{\tau}_{ij}$ as a function of the other fluid properties?)

$\longrightarrow$ Enter Navier and Stokes.



The Navier-Stokes equation for incompressible flow

~ ba-dunn ba-dunn ~

Cauchy

=

Newton's 2^{nd} law on a field

Incompressible Navier-Stokes

=

Cauchy

+

Newtonian fluid

+

incompressible flow

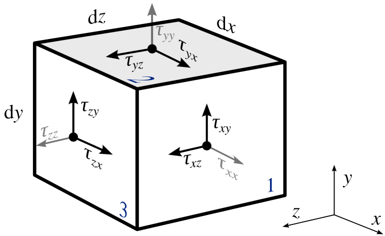
Newtonian fluid?

Then, the shear towards j on a surface perpendicular to i is

$$||\vec{\tau}_{ij}|| \;=\; \mu \frac{\partial u_j}{\partial i}$$

$\longrightarrow$ take the velocity in your direction,
find its change perpendicular to the plane.

Shear in the x-direction:



$$\vec{\tau}_{xx1} = \mu \left.\frac{\partial u}{\partial x}\right|_1 \vec{i} \qquad \vec{\tau}_{xx4} = -\mu \left.\frac{\partial u}{\partial x}\right|_4 \vec{i}$$

$$\vec{\tau}_{yx2} = \mu \left.\frac{\partial u}{\partial y}\right|_2 \vec{i} \qquad \vec{\tau}_{yx5} = -\mu \left.\frac{\partial u}{\partial y}\right|_5 \vec{i}$$

$$\vec{\tau}_{zx3} = \mu \left.\frac{\partial u}{\partial z}\right|_3 \vec{i} \qquad \vec{\tau}_{zx6} = -\mu \left.\frac{\partial u}{\partial z}\right|_6 \vec{i}$$

Shear in the *x*-direction:

The net effect, if flow incompressible:

$$\begin{aligned}
\vec{\nabla} \cdot \vec{\tau}_{ix} &= \frac{\partial \vec{\tau}_{xx}}{\partial x} + \frac{\partial \vec{\tau}_{yx}}{\partial y} + \frac{\partial \vec{\tau}_{zx}}{\partial z} \\
&= \frac{\partial \left(\mu \frac{\partial u}{\partial x} \vec{i} \right)}{\partial x} + \frac{\partial \left(\mu \frac{\partial u}{\partial y} \vec{i} \right)}{\partial y} + \frac{\partial \left(\mu \frac{\partial u}{\partial z} \vec{i} \right)}{\partial z} \\
&= \mu \left(\frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial y)^2} + \frac{\partial^2 u}{(\partial z)^2} \right) \vec{i}
\end{aligned}$$

there has to be a better way to write this!

the *Laplacian*

One more impress-your-date tool:

$$
\begin{aligned}
\vec{\nabla}^2 &\equiv \vec{\nabla} \cdot \vec{\nabla} \\
\vec{\nabla}^2 A &\equiv \vec{\nabla} \cdot \vec{\nabla} A
\end{aligned}
$$

the *Laplacian*

One more impress-your-date tool:

$$\vec{\nabla}^2\vec{A} \equiv \begin{pmatrix} \vec{\nabla}^2 A_x \\ \vec{\nabla}^2 A_y \\ \vec{\nabla}^2 A_z \end{pmatrix} = \begin{pmatrix} \vec{\nabla}\cdot\vec{\nabla}A_x \\ \vec{\nabla}\cdot\vec{\nabla}A_y \\ \vec{\nabla}\cdot\vec{\nabla}A_z \end{pmatrix}$$

Shear in the *x*-direction:

and so now:

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{\tau}_{ix} &= \mu \left(\frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial y)^2} + \frac{\partial^2 u}{(\partial z)^2} \right) \vec{i} \\
&= \mu \vec{\nabla}^2 u \, \vec{i} \\
&= \mu \vec{\nabla}^2 \vec{u}
\end{aligned}
$$

Net shear effect is change in space
of change in space of velocity

We made it!

Three components of shear force

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{\tau}_{ix} &= \mu \vec{\nabla}^2 \vec{u} \\
\vec{\nabla} \cdot \vec{\tau}_{iy} &= \mu \vec{\nabla}^2 \vec{v} \\
\vec{\nabla} \cdot \vec{\tau}_{iz} &= \mu \vec{\nabla}^2 \vec{w}
\end{aligned}
$$

Three components of shear force

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{\tau}_{ij} &= \begin{pmatrix} \vec{\nabla} \cdot \vec{\tau}_{ix} \\ \vec{\nabla} \cdot \vec{\tau}_{iy} \\ \vec{\nabla} \cdot \vec{\tau}_{iz} \end{pmatrix} = \begin{pmatrix} \mu \vec{\nabla}^2 \vec{u} \\ \mu \vec{\nabla}^2 \vec{v} \\ \mu \vec{\nabla}^2 \vec{w} \end{pmatrix} \\
\vec{\nabla} \cdot \vec{\tau}_{ij} &= \mu \vec{\nabla}^2 \vec{V}
\end{aligned}
$$

…which leads us to the glorious…

The *incompressible Navier-Stokes equation*

acceleration = gravity + pressure + shear

$$\rho \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \rho \vec{g} - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{V}$$

Describes all ordinary flow!

Question

If

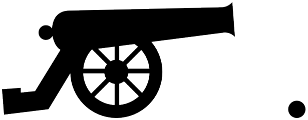
$$0 = \vec{\nabla} \cdot \vec{V}$$

$$\rho \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \rho \vec{g} - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{V}$$

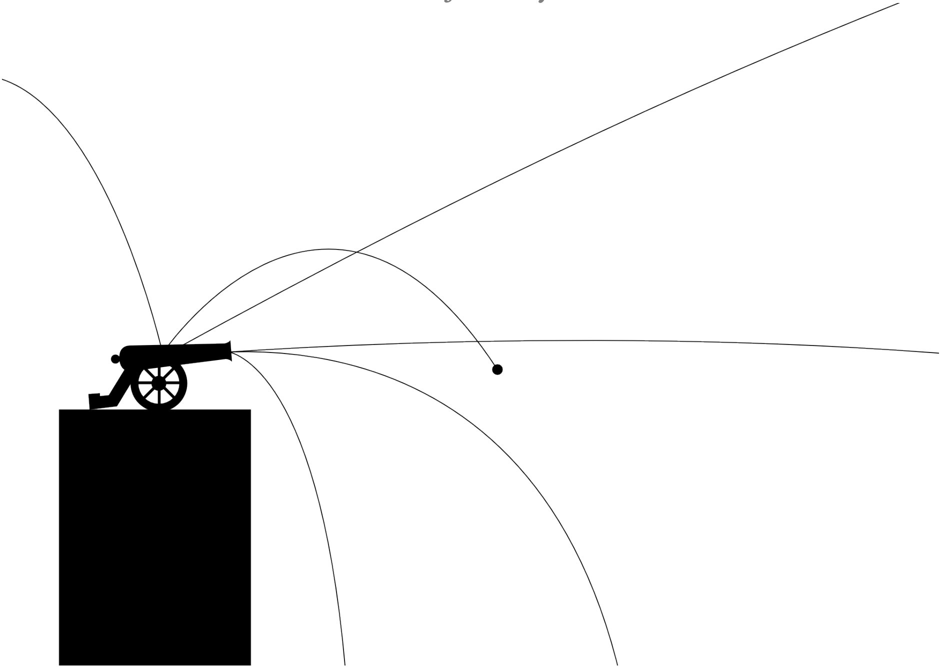
then what is $\vec{V}$?

we want *the* solution, not *a* solution

What do we mean by “the” solution?



What is the trajectory of the ball?



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What is the solution to:

$$\rho \frac{\mathrm{d}\vec{V}}{\mathrm{d}t} = \rho \vec{g}$$

“~~it depends~~”

all solutions are:

$$\vec{V} = \begin{pmatrix} u_0 \\ v_0 + gt \end{pmatrix}$$

Question

If

$$
\begin{aligned}
0 &= \vec{\nabla} \cdot \vec{V} \\
\rho \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} &= \rho\vec{g} - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{V}
\end{aligned}
$$

then what is $\vec{V}$?

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The *incompressible* *Navier-Stokes equation*

$$\rho \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \rho\vec{\mathrm{g}} - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{V}$$

In Cartesian coordinates:

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right] = \rho g_x-\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 u}{(\partial x)^2}+\frac{\partial^2 u}{(\partial y)^2}+\frac{\partial^2 u}{(\partial z)^2}\right]$$

$$\rho\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right] = \rho g_y-\frac{\partial p}{\partial y}+\mu\left[\frac{\partial^2 v}{(\partial x)^2}+\frac{\partial^2 v}{(\partial y)^2}+\frac{\partial^2 v}{(\partial z)^2}\right]$$

$$\rho\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right] = \rho g_z-\frac{\partial p}{\partial z}+\mu\left[\frac{\partial^2 w}{(\partial x)^2}+\frac{\partial^2 w}{(\partial y)^2}+\frac{\partial^2 w}{(\partial z)^2}\right]$$

Cartesian not working for you?

$$\begin{aligned}
&\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right] \\
&\quad = \quad \rho g_r - \frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{(\partial\theta)^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{(\partial z)^2}\right]
\end{aligned}$$

$$\begin{aligned}
&\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right] \\
&\quad = \quad \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{(\partial\theta)^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{(\partial z)^2}\right]
\end{aligned}$$

$$\begin{aligned}
&\rho\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right] \\
&\quad = \quad \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{(\partial\theta)^2} + \frac{\partial^2 v_z}{(\partial z)^2}\right]
\end{aligned}$$

The *incompressible Navier-Stokes equation*

$$\rho \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \rho\vec{\mathrm{g}} - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{V}$$

it's not the Bernoulli equation

What then?

Navier-Stokes helps us:

- understand and quantify the influence of forces;
- find analytical solutions sometimes (!)
- *find numerical solutions.*