Exercise sheet 4 (Differential analysis)

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These lecture notes are based on textbooks by White [12], Çengel & al.[15], and Munson & al.[17].

Except otherwise indicated, we assume that fluids are Newtonian, and that:
$\rho_{\text{water}} = 1\,000\ \text{kg m}^{-3}$; $p_{\text{atm.}} = 1\ \text{bar}$; $\rho_{\text{atm.}} = 1{,}225\ \text{kg m}^{-3}$; $T_{\text{atm.}} = 11{,}3\ °\text{C}$; $\mu_{\text{atm.}} = 1{,}5 \cdot 10^{-5}\ \text{N s m}^{-2}$;
$g = 9{,}81\ \text{m s}^{-2}$. Air is modeled as a perfect gas ($R_{\text{air}} = 287\ \text{J K}^{-1}\ \text{kg}^{-1}$; $\gamma_{\text{air}} = 1{,}4$; $c_{p\text{air}} = 1\,005\ \text{J kg}^{-1}\ \text{K}^{-1}$).

Continuity equation:

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \vec{\nabla}\cdot\vec{V} = 0 \tag{4/16}$$

Navier-Stokes equation for incompressible flow:

$$\rho\frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \rho\vec{g} - \vec{\nabla}p + \mu\vec{\nabla}^2\vec{V} \tag{4/37}$$

4.1 Revision questions

For the continuity equation (eq. 4/16), and then for the incompressible Navier-Stokes equation (eq. 4/37),

1. Write out the equation in its fully-developed form in three Cartesian coordinates;
2. State in which flow conditions the equation applies.

Also, in order to revise the notion of total (or substantial) derivative:

3. Describe a situation in which the total derivative of a property is non-zero although the fluid property is independent of time.
4. Describe a situation in which the total derivative of a property is zero although the time rate of change of this property is non-zero.

4.2 Acceleration field

Çengel & al. [15] E4-3

A flow is described with the velocity field $\vec{V} = (0{,}5 + 0{,}8x)\vec{i} + (1{,}5 - 0{,}8y)\vec{j}$ (in SI units).

What is the acceleration measured by a probe positioned at $(2; 2; 2)$ at $t = 3$ s ?

4.3 Volumetric dilatation rate

der. Munson & al. [17] 6.4

A flow is described by the following field (in SI units):

$$
\begin{aligned}
u &= x^3 + y^2 + z \\
v &= xy + yz + z^3 \\
w &= -4x^2z - z^2 + 4
\end{aligned}
$$

What is the volumetric dilatation rate field? What is the value of this rate at {2;2;2}?

4.4 Incompressibility

Çengel & al. [15] 9-28

Does the vector field $\vec{V} = (1{,}6 + 1{,}8x)\vec{i} + (1{,}5 - 1{,}8y)\vec{j}$ satisfy the continuity equation for two-dimensional incompressible flow?

4.5 Missing components

Munson & al. [17] E6.2 + Çengel & al. [15] 9-4

Two flows are described by the following fields:

$$
\begin{aligned}
u_1 &= x^2 + y^2 + z^2 \\
v_1 &= xy + yz + z \\
w_1 &= ?
\end{aligned}
$$

$$
\begin{aligned}
u_2 &= ax^2 + by^2 + cz^2 \\
v_2 &= ? \\
w_2 &= axz + byz^2
\end{aligned}
$$

What must w_1 and v_2 be so that these flows be incompressible?

4.6 Another acceleration field

White [12] E4.1

Given the velocity field $\vec{V} = (3t)\vec{i} + (xz)\vec{j} + (ty^2)\vec{k}$ (SI units), what is the acceleration field, and what is the value measured at {2;4;6} and $t = 5$ s?

4.7 Vortex

Çengel & al. [15] 9.27

A vortex is modeled with the following two-dimensional flow:

$$
\begin{aligned}
u &= C\frac{y}{x^2 + y^2} \\
v &= -C\frac{x}{x^2 + y^2}
\end{aligned}
$$

Verify that this field satisfies the continuity equation for incompressible flow.

4.8 Pressure fields

Çengel & al. [15] E9-13, White [12] 4.32 & 4.34

We consider the four (separate and independent) incompressible flows below:

$$
\begin{aligned}
\vec{V}_1 &= (ax + b)\vec{i} + (-ay + cx)\vec{j} \\
\vec{V}_2 &= (2y)\vec{i} + (8x)\vec{j} \\
\vec{V}_3 &= (ax + bt)\vec{i} + (cx^2 + ey)\vec{j} \\
\vec{V}_4 &= U_0\left(1 + \frac{x}{L}\right)\vec{i} - U_0\frac{y}{L}\vec{j}
\end{aligned}
$$

The influence of gravity is neglected on the first three fields.

Does a function exist to describe the pressure field of each of these flows, and if so, what is it?

Answers

4.1 1) For continuity, use eqs. 4/2 and 1/8 in equation 4/16. For Navier-Stokes, see eqs. 4/38, 4/39 and 4/40 p. 95; 2) Read §4.3 p. 88 for continuity, and §4.4.2 p.93 for Navier-Stokes; 3) and 4) see §4.2.2 p. 86.

4.2 $\frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = (0{,}4 + 0{,}64x)\vec{i} + (-1{,}2 + 0{,}64y)\vec{j}$. At the probe it takes the value $1{,}68\vec{i} + 0{,}08\vec{j}$ (length $1{,}682\ \mathrm{m\,s^{-2}}$).

4.3 $\vec{\nabla} \cdot \vec{V} = -x^2 + x - z$; thus at the probe it takes the value $\left(\vec{\nabla} \cdot \vec{V}\right)_{\text{probe}} = -4\ \mathrm{s^{-1}}$.

4.4 Apply equation 4/18 p.90 to $\vec{V}$: the answer is yes.

4.5 1) Applying equation 4/18: $w_1 = -3xz - \frac{1}{2}z^2 + f_{(x,y,t)}$;
2) idem, $v_2 = -3axy - bzy^2 + f_{(x,z,t)}$.

4.6 $\frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = (3)\vec{i} + (3z + y^2x)t\vec{j} + (y^2 + 2xyzt)\vec{k}$. At the probe it takes the value $3\vec{i} + 250\vec{j} + 496\vec{k}$.

4.7 Apply equation 4/18 to $\vec{V}$ to verify incompressibility.

4.8 Note: the constant (initial) value p_0 is sometimes implicitly written in the unknown functions f.
1) $p = -\rho\left[abx + \frac{1}{2}a^2x^2 + bcy + \frac{1}{2}a^2y^2\right] + p_0 + f_{(t)}$;
2) $p = -\rho\left(8x^2 + 8y^2\right) + p_0 + f_{(t)}$; 3) $\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) \neq \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)$, thus we cannot describe the pressure with a mathematical function;
4) $p = -\rho\left[\frac{U_0^2}{L}\left(x + \frac{x^2}{2L} + \frac{y^2}{2L}\right) - g_x x - g_y y\right] + p_0 + f_{(t)}$.