Exercise sheet 4 (Differential analysis)

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These lecture notes are based on textbooks by White [13], Çengel & al.[16], and Munson & al.[18].

Except otherwise indicated, we assume that fluids are Newtonian, and that:
$\rho_{\text{water}} = 1\,000\,\text{kg}\,\text{m}^{-3}$; $p_{\text{atm.}} = 1\,\text{bar}$; $\rho_{\text{atm.}} = 1{,}225\,\text{kg}\,\text{m}^{-3}$; $T_{\text{atm.}} = 11{,}3\,^{\circ}\text{C}$; $\mu_{\text{atm.}} = 1{,}5 \cdot 10^{-5}\,\text{N}\,\text{s}\,\text{m}^{-2}$;
$g = 9{,}81\,\text{m}\,\text{s}^{-2}$. Air is modeled as a perfect gas ($R_{\text{air}} = 287\,\text{J}\,\text{K}^{-1}\,\text{kg}^{-1}$; $\gamma_{\text{air}} = 1{,}4$; $c_{p\text{air}} = 1\,005\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}$).

Continuity equation:

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \vec{\nabla} \cdot \vec{V} = 0 \tag{4/16}$$

Navier-Stokes equation for incompressible flow:

$$\rho \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \rho \vec{g} - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{V} \tag{4/37}$$

4.1 Revision questions

For the continuity equation (eq. 4/16), and then for the incompressible Navier-Stokes equation (eq. 4/37),

1. Write out the equation in its fully-developed form in three Cartesian coordinates;
2. State in which flow conditions the equation applies.

Also, in order to revise the notion of total (or substantial) derivative:

3. Describe a situation in which the total time derivative $\mathrm{D}/\mathrm{D}t = 0$ of a property is non-zero, even though the flow is entirely steady ($\partial/\partial t \neq 0$).
4. Describe a situation in which the the flow is unsteady, although some property of the fluid, when measured from the point of view of the particle, is not changing with time.

4.2 Acceleration field

Çengel & al. [16] E4-3

A flow is described with the velocity field $\vec{V} = (0{,}5 + 0{,}8x)\vec{i} + (1{,}5 - 0{,}8y)\vec{j}$ (in SI units, in the laboratory frame of reference).

What is the acceleration of a particle positioned at $(2; 2; 2)$ at $t = 3\,\text{s}$?

4.3 Volumetric dilatation rate

der. Munson & al. [18] 6.4

A flow is described by the following field (in SI units):

$$u = x^3 + y^2 + z$$
$$v = xy + yz + z^3$$
$$w = -4x^2z - z^2 + 4$$

What is the volumetric dilatation rate field (the divergent of the velocity field)? What is the value of this rate at {2;2;2}?

4.4 Incompressibility

Çengel & al. [16] 9-28

Does the vector field $\vec{V} = (1{,}6 + 1{,}8x)\vec{i} + (1{,}5 - 1{,}8y)\vec{j}$ satisfy the continuity equation for two-dimensional incompressible flow?

4.5 Missing components

Munson & al. [18] E6.2 + Çengel & al. [16] 9-4

Two flows are described by the following fields:

$$u_1 = x^2 + y^2 + z^2$$
$$v_1 = xy + yz + z$$
$$w_1 = ?$$

$$u_2 = ax^2 + by^2 + cz^2$$
$$v_2 = ?$$
$$w_2 = axz + byz^2$$

What must w_1 and v_2 be so that these flows be incompressible?

4.6 Another acceleration field

White [13] E4.1

Given the velocity field $\vec{V} = (3t)\vec{i} + (xz)\vec{j} + (ty^2)\vec{k}$ (SI units), what is the acceleration field, and what is the value measured at {2;4;6} and $t = 5$ s?

4.7 Vortex

Çengel & al. [16] 9.27

A vortex is modeled with the following two-dimensional flow:

$$u = C\frac{y}{x^2 + y^2}$$
$$v = -C\frac{x}{x^2 + y^2}$$

Verify that this field satisfies the continuity equation for incompressible flow.

4.8 Pressure fields

Çengel & al. [16] E9-13, White [13] 4.32 & 4.34

We consider the four (separate and independent) incompressible flows below:

$$\begin{aligned}
\vec{V}_1 &= (ax + b)\vec{i} + (-ay + cx)\vec{j} \\
\vec{V}_2 &= (2y)\vec{i} + (8x)\vec{j} \\
\vec{V}_3 &= (ax + bt)\vec{i} + (cx^2 + ey)\vec{j} \\
\vec{V}_4 &= U_0 \left(1 + \frac{x}{L}\right)\vec{i} - U_0 \frac{y}{L}\vec{j}
\end{aligned}$$

The influence of gravity is neglected on the first three fields.

Does a function exist to describe the pressure field of each of these flows, and if so, what is it?

Answers

4.1 1) For continuity, use eqs. 4/2 and 1/8 in equation 4/16. For Navier-Stokes, see eqs. 4/38, 4/39 and 4/40 p. 96; 2) Read §4.3 p. 89 for continuity, and §4.4.2 p.94 for Navier-Stokes; 3) and 4) see §4.2.2 p. 86.

4.2 $\frac{D\vec{V}}{Dt} = (0{,}4 + 0{,}64x)\vec{i} + (-1{,}2 + 0{,}64y)\vec{j}$. At the probe it takes the value $1{,}68\vec{i} + 0{,}08\vec{j}$ (length $1{,}682\ \mathrm{m\,s^{-2}}$).

4.3 $\vec{\nabla} \cdot \vec{V} = -x^2 + x - z$; thus at the probe it takes the value $\left(\vec{\nabla} \cdot \vec{V}\right)_{\text{probe}} = -4\ \mathrm{s^{-1}}$.

4.4 Apply equation 4/18 p.91 to $\vec{V}$: the answer is yes.

4.5 1) Applying equation 4/18: $w_1 = -3xz - \frac{1}{2}z^2 + f_{(x,y,t)}$;
2) idem, $v_2 = -3axy - bzy^2 + f_{(x,z,t)}$.

4.6 $\frac{D\vec{V}}{Dt} = (3)\vec{i} + (3z + y^2x)t\vec{j} + (y^2 + 2xyzt)\vec{k}$. At the probe it takes the value $3\vec{i} + 250\vec{j} + 496\vec{k}$.

4.7 Apply equation 4/18 to $\vec{V}$ to verify incompressibility.

4.8 Note: the constant (initial) value p_0 is sometimes implicitly written in the unknown functions f.
1) $p = -\rho\left[abx + \frac{1}{2}a^2x^2 + bcy + \frac{1}{2}a^2y^2\right] + p_0 + f_{(t)}$;
2) $p = -\rho\left(8x^2 + 8y^2\right) + p_0 + f_{(t)}$; 3) $\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) \neq \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)$, thus we cannot describe the pressure with a mathematical function;
4) $p = -\rho\left[\frac{U_0^2}{L}\left(x + \frac{x^2}{2L} + \frac{y^2}{2L}\right) - g_x x - g_y y\right] + p_0 + f_{(t)}$.