Hello! You are consulting an examination paper from the archives at https://fluidmech.ninja/.

In the summer semester 2020, the general structure of the examination will be largely the same as in this archive. Nevertheless, because the course content progressively changes from year to year, there are a few differences. In former years,

- The course contained a chapter about compressible air flow (involving tables for air properties) that is no longer part of the course now;
- Conversely, several chapters have been added to the examinable content over the years;
- The course contained a duct flow problem involving a ball fountain (“Kugel fountain”) that is no longer part of the course now;
- Viscosity values were read in a different diagram, and may not match values read in the 2020 viscosity diagram;
- Many small updates in the notation had not yet been carried out.

To obtain precise information about the summer semester 2020 examination, consult the dedicated appendix in the lecture notes. If you have questions, contact me as detailed in the course syllabus. Thanks, and good luck in your revisions!

Olivier Cleynen
September 2020
Fluid Mechanics examination — July 11, 2019

Fluid Mechanics for Master Students

Solve problem 1, plus three other problems among problems 2 to 6.

Duration: 2h – Use of calculator is authorized; documents are not authorized.

Except otherwise indicated, we assume that:

Fluids are Newtonian

The atmosphere has $\rho_{\text{atm.}} = 1 \text{ bar}; \; T_{\text{atm.}} = 11.3 \, ^\circ\text{C}; \; \mu_{\text{atm.}} = 1.5 \cdot 10^{-3} \text{ Pa s}$

Air behaves as a perfect gas: $R_{\text{air}} = 287 \, \text{J kg}^{-1} \text{K}^{-1}; \; \gamma_{\text{air}} = 1.4; \; c_{p_{\text{air}}} = 1005 \, \text{J kg}^{-1} \text{K}^{-1}; \; c_{\rho_{\text{air}}} = 718 \, \text{J kg}^{-1} \text{K}^{-1}$

Liquid water is incompressible: $\rho_{\text{water}} = 1000 \, \text{kg m}^{-3}, \; c_{p_{\text{water}}} = 4180 \, \text{J kg}^{-1} \text{K}^{-1}$

Balance of mass in a considered volume with steady flow:

$$0 = \sum [\rho V_i A]_{\text{incoming}} + \sum [\rho V_i A]_{\text{outgoing}}$$

where $V_i$ is negative inwards, positive outwards.

Balance of momentum in a considered volume with steady flow:

$$\vec{F}_{\text{net on fluid}} = \sum [\rho V_i A \vec{V}]_{\text{incoming}} + \sum [\rho V_i A \vec{V}]_{\text{outgoing}}$$

where $V_i$ is negative inwards, positive outwards.

Balance of energy in a considered volume with steady flow:

$$\dot{Q}_{\text{net}} + \dot{W}_{\text{shaft, net}} = \sum \left[ \dot{m} \left( i + \frac{p}{\rho} + \frac{1}{2} V^2 + gz \right) \right]_{\text{in}}$$

$$+ \sum \left[ \dot{m} \left( i + \frac{p}{\rho} + \frac{1}{2} V^2 + gz \right) \right]_{\text{out}}$$

where $\dot{m}$ is negative inwards, positive outwards.

Mass balance through an arbitrary volume:

$$0 = \frac{d}{dt} \iiint_{CV} \rho \, dV + \iint_{CS} \rho \, (\vec{V}_{\text{rel}} \cdot \hat{n}) \, dA$$

Momentum balance through an arbitrary volume:

$$\vec{F}_{\text{net}} = \frac{d}{dt} \iiint_{CV} \rho \vec{V} \, dV + \iint_{CS} \rho \vec{V} \, (\vec{V}_{\text{rel}} \cdot \hat{n}) \, dA$$

Angular momentum balance through an arbitrary volume:

$$\vec{M}_{\text{net},X} = \frac{d}{dt} \iiint_{CV} \vec{r}_{Xm} \wedge \rho \vec{V} \, dV + \iint_{CS} \vec{r}_{Xm} \wedge \rho \, (\vec{V}_{\text{rel}} \cdot \hat{n}) \vec{V} \, dA$$
Continuity equation for incompressible flow:
\[ \vec{V} \cdot \vec{V} = 0 \]  
(7)

Navier-Stokes equation for incompressible flow:
\[ \rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \vec{V} p + \mu \vec{V}^2 \vec{V} \]  
(8)

In a highly-viscous (creeping) steady flow, the drag \( F_D \) exerted on a spherical body of diameter \( D \) at by flow at velocity \( V_\infty \) is quantified as:
\[ F_{D_{sphere}} = 3\pi \mu V_\infty D \]  
(9)

In cylindrical pipe flow, we assume the flow is always laminar for [Re]_D \( \lesssim 2300 \), and always turbulent for [Re]_D \( \gtrsim 4000 \). The Darcy friction factor \( f \) is defined as:
\[ f = \frac{|\Delta p_{\text{loss}}|}{\frac{4}{3} \rho V^2_{av.}} \]  
(10)

The loss coefficient \( K_L \) is defined as:
\[ K_L = \frac{|\Delta p_{\text{loss}}|}{\frac{1}{2} \rho V^2_{av.}} \]  
(11)

Viscosities of various fluids are given in fig. 1. Pressure losses in cylindrical pipes can be calculated with the help of the Moody diagram presented in fig. 2 p.4.

Non-dimensional incompressible Navier-Stokes equation:
\[ [St] \frac{\partial \vec{V}^*}{\partial t^*} + [1] \vec{V}^* : \vec{\nabla} \vec{V}^* = \frac{1}{[Fr]^2} \vec{g}^* - [Eu] \vec{\nabla} p^* + \frac{1}{[Re]} \vec{V}^*^2 \vec{V}^* \]  
(12)
in which [St] = \( \frac{L}{V^*} \), [Eu] = \( \frac{\rho_1 - \rho_0}{\rho \sqrt{V^*}} \), [Fr] = \( \frac{V}{\sqrt{g L}} \) and [Re] = \( \frac{\rho V L}{\mu} \).

The force coefficient \( C_F \) and power coefficient \( C_P \) are defined as:
\[ C_F = \frac{F}{\frac{1}{2} \rho SV^2} \quad \quad C_P = \frac{W}{\frac{1}{2} \rho SV^3} \]  
(13)

The speed of sound \( c \) in air is modeled as:
\[ c = \sqrt{\gamma RT} \]  
(14)
In boundary layer flow, we assume that transition occurs at \( \text{Re}_x \gtrsim 5 \cdot 10^5 \).

The wall shear coefficient \( c_f \), a function of distance \( x \), is defined based on the free-stream flow velocity \( U \):

\[
c_f = \frac{\tau_{\text{wall}}}{\frac{1}{2} \rho U^2}
\]  

(15)

Exact solutions to the laminar boundary layer along a smooth surface yield:

\[
\frac{\delta}{x} = \frac{4.91}{\sqrt{\text{Re}_x}} \quad \quad \frac{\delta^*}{x} = \frac{1.72}{\sqrt{\text{Re}_x}}
\]  

(16)

\[
\frac{\delta^*}{x} = \frac{0.664}{\sqrt{\text{Re}_x}} \quad \quad c_f = \frac{0.664}{\sqrt{\text{Re}_x}}
\]  

(17)

Solutions to the turbulent boundary layer along a smooth surface yield the following time-averaged characteristics:

\[
\frac{\delta}{x} \approx \frac{0.16}{[\text{Re}_x]^\frac{1}{2}} \quad \quad \frac{\delta^*}{x} \approx \frac{0.02}{[\text{Re}_x]^\frac{1}{2}}
\]  

(18)

\[
\frac{\delta^*}{x} \approx \frac{0.016}{[\text{Re}_x]^\frac{1}{4}} \quad \quad c_f \approx \frac{0.027}{[\text{Re}_x]^\frac{1}{4}}
\]  

(19)

In a highly-viscous (creeping) steady flow, the drag \( F_D \) exerted on a spherical body of diameter \( D \) at by flow at velocity \( U_\infty \) is quantified as:

\[
F_{D, \text{sphere}} = 3 \pi \mu U_\infty D
\]  

(20)

Figure 1 – Viscosity of various fluids at a pressure of 1 bar (in practice viscosity is almost independent of pressure).
Figure 2 – A Moody diagram, which presents values for $f$ measured experimentally, as a function of the diameter-based Reynolds number $[\text{Re}]_D$, for different relative roughness values.

Diagram CC-BY-SA S Beck and R Collins, University of Sheffield
Solve problem 1, 
and three other problems among problems 2 to 6.

The following marking guidelines will be used:

- Answers to questions starting with “show that” should be fully-developed and continuous;
- In all other questions, the correct result with the correct unit is enough to obtain full points;
- Illegible or ambiguous answers are always discarded.
1 Governing equation

1.1. [5 pts] Write out equation (8), the Navier-Stokes equation for incompressible flow, in its fully-developed form in three Cartesian coordinates.

1.2. [5 pts] Write out equation (7), the continuity equation for incompressible flow, in its fully-developed form in three Cartesian coordinates.

2 Observation window in a water tank

A water tank used in a laboratory is filled with stationary water (fig. 3). A window is installed on one of the walls of the canal, to enable observation. The window is hinged on its top face.

The window has a height of 1.5 m and a width of 3.5 m. The walls of the tank are inclined with an angle $\theta = 70^\circ$ relative to horizontal.

2.1. [15 pts] What is the magnitude of the net force applying on the tank window?

2.2. [10 pts] At what distance away from the hinge does this force apply?

Water is added to the tank, so that the water level increases.

2.3. [5 pts] How will the distance calculated above change as water is added? (briefly justify your answer, e.g. in 30 words or less)
3 Piping leading to a turbine

A pipe leads water from one reservoir to a turbine, which discharges into another reservoir, as shown in figure 4.

![Figure 4 – Layout of the water pipe. For clarity, in this figure, the vertical scale is greatly exaggerated. In the vertical scale, the diameter of the pipe is also greatly exaggerated.](image)

The pipe is made of coarse concrete (roughness 0.25 mm) and carries 800 L s\(^{-1}\) of water at 20 °C. It has a diameter of 1.1 m and features four elbow bends with sharp angles, each inducing a loss coefficient \(K_L\) of 0.75.

3.1. [10 pts] Represent qualitatively (i.e. without numerical data) the pressure distribution along the length of the pipe, both when the turbine is shut down (without any flow), and when it is operating.

3.2. [15 pts] What is the hydraulic power available to the turbine?

The outlet tank on the right is very large, so that its water level does not vary. The source water tank on the left, however, sees its height decrease as the water is emptied through the turbine. Ultimately, as the water level decreases, the water stops flowing entirely.

3.3. [5 pts] When the water stops flowing, what will be the height of the water level in the source tank on the left?
4 Boundary layer on a flat plate

A thin and smooth plate with width $W = 0.6\,\text{m}$ and length $L = 2\,\text{m}$ is placed with a zero angle of attack in atmospheric air flow incoming at $21\,\text{m\ s}^{-1}$, as shown in figure 5. We would like to study the shear exerted by the flow over the top surface of the plate.

Figure 5 – A thin plate positioned parallel to an incoming uniform flow.

4.1. [5 pts] At what distance $x_{tr}$ along the plate, approximately, will the boundary layer transit and become turbulent?

4.2. [10 pts] Starting from equation (21), which quantifies the friction factor $c_f$ (see definition 15) in a laminar boundary layer,

$$c_f = \frac{0.664}{\sqrt{Re_x}}$$

(21)

show that the shear force $F_{r \text{ laminar}}$ exerted in the laminar section of the boundary layer is:

$$F_{r \text{ laminar}} = 0.664 \sqrt{\mu} U^{\frac{1}{2}} W x_{tr}^{\frac{1}{2}}$$

(22)

4.3. [5 pts] What is the shear force exerted on the top surface of the plate by the laminar section of the boundary layer?

4.4. [5 pts] What is the shear force exerted on the top surface of the plate by the turbulent section of the boundary layer?

4.5. [5 pts] Would the boundary layer become thicker if the velocity was increased? (briefly justify your answer, e.g. in 30 words or less).
5 Velocity measurements in a tunnel

A group of students proceeds with speed measurements in a water tunnel. The objective is to measure the drag applying on an object with constant cross-section, positioned across the tunnel test section (fig. 6).

![Diagram of velocity measurements in a tunnel](image)

Figure 6 – An object with constant cross-section positioned across a water tunnel. The object spans completely across the tunnel (in the z-direction). The horizontal velocity distributions upstream and downstream of the profile are also shown.

Upstream of the object, the water flow velocity is uniform \((u_1 = U = 3.2 \text{ m s}^{-1})\).

Downstream of the object, horizontal velocity measurements are made every 5 cm across the flow; the following results are obtained:

<table>
<thead>
<tr>
<th>vertical position (y) (cm)</th>
<th>horizontal speed (u_2) (m s(^{-1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.2</td>
</tr>
<tr>
<td>5</td>
<td>3.2</td>
</tr>
<tr>
<td>10</td>
<td>3.15</td>
</tr>
<tr>
<td>15</td>
<td>3.14</td>
</tr>
<tr>
<td>20</td>
<td>3.03</td>
</tr>
<tr>
<td>25</td>
<td>2.92</td>
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<tr>
<td>30</td>
<td>2.81</td>
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<td>35</td>
<td>2.87</td>
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<td>2.89</td>
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<td>45</td>
<td>2.97</td>
</tr>
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<td>50</td>
<td>3.19</td>
</tr>
<tr>
<td>55</td>
<td>3.2</td>
</tr>
<tr>
<td>60</td>
<td>3.2</td>
</tr>
</tbody>
</table>
The width of the profile (perpendicular to the flow, in the $z$-direction) is 70 cm. The water has uniform temperature and density ($20^\circ C, 999\, \text{kg} \cdot \text{m}^{-3}$) and the pressure is uniform across the measurement surface.

5.1. [20 pts] What is the drag force applying on the profile?

5.2. [10 pts] If water was replaced with a fluid with higher viscosity, how would you expect the drag force to change? (briefly justify your answer, e.g. in 30 words or less)

6 Lift and drag on a rotating football

A group of fluid dynamicists investigates the air flow around a football. In particular, they are interested in the forces applying on the ball when it has been kicked and is flying through the air. The football has diameter 22 cm, a weight of 430 g; it is traveling at 70 km h$^{-1}$.

In order to observe the flow, they install a steel sphere in a wind tunnel (figure 7). The sphere has a diameter of 1.1 m. Drag force measurements are carried out in the tunnel.

![Figure 7 – A steel sphere positioned in a wind tunnel. The sphere is maintained stationary, while the air travels with speed $V_{\text{tunnel}}$. Force measurements are carried out on the ball.](image)

*Figure CC-0 o.c.*
6.1. [5 pts] What is the wind tunnel speed required, so that the flow around the real football is reproduced around the sphere in the tunnel?

6.2. [5 pts] With the speed calculated above, by which factor should the drag force measured in the wind tunnel be multiplied, in order to obtain the drag force on the real football?

The fluid dynamicists now investigate the effect of spin on the ball. When the football is rotated along a horizontal axis during travel, a lift force exerts laterally on the ball, curving its trajectory. This is represented, from above, in figure 8.

In order to quantify this effect, the wind tunnel sphere is rotated in the wind tunnel, and measurements are carried out; the results are plotted in figure 9.

Figure 8 – Trajectory of a rotating football in free flight, as seen from above. A lift force exerts towards the left, and deviates the trajectory towards the left.

Figure CC-0 o.c.
6.3. [10 pts] How many rotations per second are required in order to generate a lift force of 3.1 N on the real football when it travels?

6.4. [5 pts] What is then the corresponding drag force?

6.5. [5 pts] Propose and quantify one possibility for the football player to double the lift force applying on the ball.