Hello! You are consulting an examination paper from the archives at https://fluidmech.ninja/.

In the summer semester 2020, the general structure of the examination will be largely the same as in this archive. Nevertheless, because the course content progressively changes from year to year, there are a few differences. In former years,

- The course contained a chapter about compressible air flow (involving tables for air properties) that is no longer part of the course now;

- Conversely, several chapters have been added to the examinable content over the years;

- The course contained a duct flow problem involving a ball fountain (“Kugel fountain”) that is no longer part of the course now;

- Viscosity values were read in a different diagram, and may not match values read in the 2020 viscosity diagram;

- Many small updates in the notation had not yet been carried out.

To obtain precise information about the summer semester 2020 examination, consult the dedicated appendix in the lecture notes. If you have questions, contact me as detailed in the course syllabus. Thanks, and good luck in your revisions!

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September 2020
1. \[
\rho \left[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] = \rho g x - \frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right]
\]
\[
\rho \left[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right] = \rho g y - \frac{\partial p}{\partial y} + \mu \left[ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right]
\]
\[
\rho \left[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right] = \rho g z - \frac{\partial p}{\partial z} + \mu \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right]
\]

2. For all flows that are incompressible and in which the fluid is Newtonian.
2.1) \[ \nabla p = \rho g \]

With \( z \) oriented vertically downwards: \[ T \int_0^z \rho \, dz \]

\( g = g_z \) and \( \frac{dp}{dz} = \frac{dp}{dz} \)

\[ \frac{dp}{dz} = \rho g \]

If air is modeled as a perfect gas, then \( \rho = \frac{p}{RT} \)

\[ \frac{dp}{dz} = \frac{p}{RT} g \]

\[ dp \frac{1}{p} = \frac{g}{RT} \, dz \]

If \( T = T_{\text{ref}} \) independent of \( p \) and \( z \),

\[ \int_0^z \frac{1}{T} \, dz = \int_0^z \frac{g}{RT_{\text{ref}}} \, dz \]

\[ P_0 \frac{P_e}{P_1} = \frac{g}{RT_{\text{ref}}} (\Delta z) \]

\[ \frac{P_2}{\rho} = \exp \left( \frac{g \Delta z}{RT_{\text{ref}}} \right) \]
\[ \Delta z = 0.15 \text{ m} \]

\[ T_{\text{est}} = 28^\circ \text{C} = 301.15 \text{ K} \]

\[ \frac{p_2}{p_1} = \exp \left( \frac{9.51 \times 0.15}{887 \times 301.15} \right) \]

\[ \frac{p_2}{p_1} = 1 + 1.7025 \times 10^{-5} \]

If \( p_1 = 1 \text{ bar} \), then \( p_2 - p_1 = 1.7025 \times 10^{-5} \times 10^3 \text{ Pa} \)

\[ p_2 - p_1 = \Delta p = 1.7025 \text{ Pa} \]

\[ F_{\text{buoyancy}} = \Delta p \cdot S_{\text{hazard}} = 1.7025 \times \pi \times \frac{0.12^2}{4} \]

\[ F_{\text{buoyancy}} = 1.926 \times 10^{-2} \text{ N} \text{ upwards (very small force!)} \]
2.3) In water, \( \nabla p = \rho g \) becomes

\[
\frac{\Delta p}{\Delta z} = \rho g \quad \text{since} \quad \rho = \text{constant}
\]

\[
\Delta p = \rho g \Delta z = 10^3 \times 9.81 \times 0.15
\]

\[
\Delta p = p_2 - p_1 = 1471.5 \text{ Pa}
\]

\[
F_{\text{buoyancy}} = \Delta p S_{\text{horizontal}}
\]

\[
F_{\text{buoyancy}} = 1664 \text{ N} \quad \text{upwards (approx. 1000 times more)}
\]

2.4) If the box is rotated, the force will remain constant. This is because \( \rho_{\text{water}} \) is uniform. The volume displaced by the box does not change, and so neither does the weight of water displaced, and so the buoyancy force.
3) Friction drag due to a boundary layer

\[ V = 110 \text{ km/h} = 30.56 \text{ m/s} \]

\[ R_e = 1 \text{ m} \]

\[ L = 15 \text{ m} \]

\[ T = 5^\circ \text{C} \]

\[ \mu = 1.8 \times 10^{-5} \text{ Pa.s} \quad \text{from Fig. 2} \]

\[ \rho = 1.2 \text{ kg/m}^3 \]

3.1) We search for the position \( x_{1r} \) of the BL transition:

\[ [Re]_{x_{1r}} = 5 \times 10^5 = \frac{L}{\rho U} x_{1r} \]

\[ x_{1r} = 5 \times 10^5 \frac{x_{1r}}{\rho U} \]

\[ x_{1r} = 5 \times 10^5 \frac{x_{1r}}{1.2 \times 30.56} \]

\[ x_{1r} = 0.2455 \text{ m} \]
At \( x_\Theta = 0.1 \text{ m} \), the boundary layer is still laminar.

From eq. (11)

\[
\frac{\delta}{x_\Theta} = \frac{4.81}{\left[ \frac{R_\infty x_\Theta}{1} \right]^{1/2}}
\]

\[
\delta = \frac{4.81 \times 0.1}{\left[ \frac{1/2 \times 3036 \times 0.1}{1.8 \times 10^{-5}} \right]^{1/2}}
\]

\[
\delta = 1.0879 \times 10^{-3} \text{ m} \quad (0.1 \text{ mm})
\]

3.2) Friction force on the wing surface:

\[
F = \int_S \tau \, dS = \int_0^x \tau \, L \, dx
\]

\[
= \int_0^x \frac{1}{2} \rho U^2 c_f \, L \, dx = \frac{1}{2} \rho U^2 L \int_0^x c_f \, dx
\]
In the laminar portion:

\[ F_{\text{lem}} = \frac{1}{2} \rho \frac{u^2 L}{2} \int_0^{x_{tr}} 0.664 \left( \frac{\rho u x}{\mu} \right)^{-\frac{1}{2}} dx \]

\[ = 0.332 \rho \frac{1}{2} U \frac{3}{2} \mu \frac{1}{2} L \int_0^{x_{tr}} x^{-\frac{1}{2}} dx \]

\[ = 0.332 \rho U \frac{1}{2} U \frac{3}{2} \mu \frac{1}{2} L \left[ \frac{1}{-\frac{1}{2}+1} \right]^{x_{tr}} \]

\[ F_{\text{lem}} = 0.164 \rho \frac{1}{2} U \frac{3}{2} \mu \frac{1}{2} L \left[ x^{0.5} \right]^{x_{tr}} \]

In the turbulent portion:

\[ F_{\text{turb}} = \frac{1}{2} \rho U^2 L \int_{x_{tr}}^{x_{\text{max}}} 0.087 \left[ \frac{\rho u x}{\mu} \right]^{-\frac{1}{2}} dx \]

\[ = 0.0135 \rho \frac{1}{2} U \frac{3}{2} \mu \frac{1}{2} L \left[ \frac{1}{-\frac{1}{2}+1} \right]^{x_{\text{max}}} \]

\[ F_{\text{turb}} = 0.01575 \rho \frac{1}{2} U \frac{3}{2} \mu \frac{1}{2} L \left[ x^{\frac{6}{4}} \right]^{x_{\text{max}}} \]

\[ x_{\text{max}} \]
For one slope, we have:

\[ F_{\text{lam}} = 0.664 \times 1.2 \times 30.56 \times (1.8 \times 10^{-5})^{\frac{1}{2}} \times 15 \times (0.2455) \]

\[ F_{\text{lam}} = 3.8739 \text{ N} \]

\[ F_{\text{hub}} = 0.01575 \times 1.2 \times 30.56 \times (1.8 \times 10^{-7})^{\frac{1}{2}} \times 15 \times \left(1 - 0.8455^{6/2}\right) \]

\[ F_{\text{hub}} = 83.8551 \text{ N} \]

\[ F_{\text{total}} = F_{\text{lam}} + F_{\text{hub}} = 87.13 \text{ N} \]

\[ W_{\text{friction}} = 4 \times F_{\text{total}} \times u \]

\[ W_{\text{friction}} = 4 \times 87.13 \times 30.56 \]

\[ W_{\text{friction}} = 3.316 \text{ kW} \]
3.4) If the BL was fully-laminar, we would use only eq. (A):

\[ F_{\text{New \ 2}} = F_{\text{Old \ 1}} \frac{1}{\sqrt{0.9455^{1/2}}} \]

\[ F_{\text{New \ 2}} = 7,8185 \text{ N} \]

\[ \frac{F_{\text{New \ 2}}}{F_{\text{Old \ 1}}} = \frac{7,8185}{27,13} = 0.288 \]

\[ \text{Let a 71.3% drag reduction.} \]

3.5) Laminar BLs are more likely to separate. This would restrict the flight domain of the aircraft, especially at high lift coefficients (low speed, strong adverse pressure gradients on top surface).
4) Water turbine blade

\[ V_{\text{jet}} = 35 \text{ m/s} \]
\[ D_1 = 10 \text{ cm} \]
\[ S_1 = \frac{\pi D_1^2}{4} = \frac{\pi \times 0.1^2}{4} = 7.854 \times 10^{-3} \text{ m}^2 = S_1 \]
\[ \rho = 10^3 \text{ kg/m}^3 \]

4.1) No friction: \( \| V_1' \| = \| V_2' \| \)

No blade normal: \( V_1 = V_{\text{jet}} - V_{\text{blade}} = V_{\text{jet}} \)
\[ = 0 \text{ (steady, slow)} \]

\[ \vec{F}_{\text{net}} = \frac{d}{dt} \iint_V \rho \vec{V} \, dV + \oint_{CS} \rho \vec{V} \cdot (\vec{V}_\perp) \, dA \]
\[ F_{\text{jet}, x} = - \dot{m} \ V_{1x} + \dot{m} \ V_{2x} \]
\[ = \dot{m} \left( - \frac{\overrightarrow{V}_2 \cdot \cos \Theta}{\left| \overrightarrow{V}_2 \right|} - \frac{\overrightarrow{V}_1}{\left| \overrightarrow{V}_1 \right|} \right) \]
\[ = \dot{m} \left( \frac{\left| \overrightarrow{V}_1 \right|}{- \cos \Theta - 1} \right) \]

\[ F_{\text{jet}, x} = (- \cos \Theta - 1) \ \rho \ S_1 \ V_1^2 \]

\[ = (- \cos 25 - 1) \times 1.0^3 \times 7.854 \times 10^{-3} \times 35^2 \]

\[ F_{\text{jet}, x, 4.1} = -18,341 \ \text{KN} \quad \text{(force towards the left)} \]

4.2) \[ \dot{W}_{\text{blade, 4.2}} = \left| F_{\text{jet}, x} \right| \ V_{\text{blade}} = \begin{bmatrix} 0 \ \text{W} = \dot{W}_{\text{blade, 4.2}} \end{bmatrix} \]

4.3) \[ \text{Now } V_1 = V_{\text{jet}} - V_{\text{blade}} = 0 \ \text{m/s} = V_1 \]

So \[ F_{\text{jet}, x} = 0 \ \text{N} \]

4.4) \[ \dot{W}_{\text{blade, 4.4}} = \left| F_{\text{jet}, x} \right| \ V_{\text{blade}} = \begin{bmatrix} 0 \ \text{W} = \dot{W}_{\text{blade, 4.4}} \end{bmatrix} \]
4.5)

\[ V_{\text{blade}} = 25 \text{ m/s} \]

so now \[ V_1 = V_{\text{jet}} - V_{\text{blade}} = 10 \text{ m/s} = V_1 \]

\[ F_{\text{jet}x} = (-\cos 85 - 1) \times 10^3 \times 7,854 \times 10^{-2} \times 10^6 \]

\[ F_{\text{jet}x,4.5} = -1,497 \text{ kN} \]

\[ \dot{\mathbf{W}}_{\text{blade}} = \| F_{\text{jet}x} \| \cdot \dot{V}_{\text{blade}} = \left| -1,497 \times 10^3 \right| \times 85 \]

\[ \dot{\mathbf{W}}_{\text{jet}+\text{blade},4.5} = 37,43 \text{ kW} \]

4.6)

\[ \dot{\mathbf{W}}_{\text{blade}} = (\cos \Theta - 1) \rho S_1 \left( V_{\text{jet}} - V_{\text{blade}} \right)^2 \times \dot{V}_{\text{blade}} \]

\[ \dot{\mathbf{W}}_{\text{blade}} = \rho S_1 (\cos \Theta - 1) \left[ \dot{V}_{\text{blade}}^3 + \dot{V}_{\text{jet}}^3 - 2 \dot{V}_{\text{jet}} \dot{V}_{\text{blade}}^2 + \dot{V}_{\text{jet}}^2 \dot{V}_{\text{blade}} \right] \]

extremum: \[ V_{\text{blade}} = V_{\text{blade opt}} \] \[ \text{when } \frac{\partial \dot{\mathbf{W}}_{\text{blade}}}{\partial V_{\text{blade}}} = 0 \]
\[ \frac{\partial W_{\text{blade}}}{\partial V_{\text{blade}}} = 0 \]

\[ 3 V_{\text{blade}}^2 - 4 V_{\text{jet}} V_{\text{blade}} + V_{\text{jet}}^2 = 0 \]

\[ V_{\text{blade}} = \frac{1}{3} V_{\text{jet}} \quad \text{or} \quad V_{\text{blade}} = V_{\text{jet}} \]

\[ \text{max power} \quad \text{or} \quad \text{zero power} \]
5. Flow in a converging-diverging nozzle

\[
\begin{align*}
P_0 & \\
T_0 & \\
A_0 &= \infty \\
\end{align*}
\]

\[
\begin{align*}
p_1 &= 560 \text{ kPa} \\
v_1 &= 180 \text{ m/s} \\
T_1 &= 488 \text{ K} \\
A_1 &= 0.1 \text{ m}^2 \\
A_2 &= 0.05 \text{ m}^2 \\
A_3 &= 0.835 \text{ m}^2 \\
\end{align*}
\]

5.1) \[
T_0 = T_1 + \frac{1}{c_p} \frac{1}{2} v_1^2 \\
= 488 + \frac{1}{1005} \frac{1}{2} 180^2 \\
= 445.96 \text{ K}
\]
\[ \dot{m} = \dot{m}_1 = \rho_1 \frac{V_1 A_1}{R T_1} \]

\[ = \frac{560 \times 10^3}{287 \times 428} \times 190 \times 0.1 \]

\[ \dot{m} = 86.62 \text{ kg/s} \]

\[ [\Pi a]_1 = \frac{V_1}{a_1} = \frac{V_1}{\sqrt[3]{Y RT_1}} = 190 \times \left[ 1.4 \times 287 \times 428 \right]^{-\frac{1}{2}} \]

\[ [\Pi a]_1 = 0.4582 \]

Interpolating in Fig. 3, we get

\[ \frac{P_1}{P_0} = 0.8856 + 0.582 \times (0.8480 - 0.8856) \]

\[ P_0 = \frac{P_1}{0.865} = \frac{560 \times 10^3}{0.865} = 647.4 \times 10^3 \text{ Pa} = P_0 \]
We have \( \frac{P_i}{P_0} = 0.865 \) and \([Na]_1 = 0.4562\).

Per figure 5, we need \( \frac{A_1}{A^*} \geq 1.34 \).

So
\[
A^* = A_1 \frac{1}{1.34}
\]

\[
A^* \approx 0.0746 \text{ m}^2
\]

This is the value of \( A_1 \) required so that the throat flow may be sonic.
5.3) Two possibilities:

A) Flow is always subsonic

Then in \( \text{Eq. } 2 \) the pressure returns to the value at (1), i.e., \( p_2 = p_1 = 560 \text{ kPa} \)

B) Flow becomes supersonic

Here in \( \text{Eq. } 2 \) we have \( 1.6 < \gamma < 1.8 \)

and \( 0.9353 < \frac{p_2}{p_0} < 0.9474 \) \( \rightarrow \) by interpolation

\( p_2 = 0.9047 \ p_0 \)

\( p_2 = 114.6 \text{ kPa} \)

(much lower)
5.4) We have \[ \frac{A_3}{A^*} = \frac{0.1216}{0.0746} = 1.635 \]

From fig 3, if there is to be no shock,
\[ p_3 = 2.6 \]

to generate this:

\[ \frac{P_3}{P_0} = \frac{P_2}{P_0} = 0.0501 \]

\[ P_0 \text{ must be raised to } P_0 = \frac{1}{0.0501} \times 1146 \times 10^3 \]

\[ P_0 = 22,87 \text{ bar} \]

\[ \rightarrow \text{ Increases in } P \text{, } T \text{, or a decrease of } A_3 \text{ are also possible} \]
6) Ducting for a water turbine

\[ L = 8 \times 10^3 \text{ m} \]

\[ D = 2 \text{ m} \]

\[ \varepsilon = 0.1 \text{ mm} \]

\[ \dot{V} = 5 \text{ m}^3/\text{s} \]

\[ \mu_{\text{water}} = 10^{-3} \text{ Pa} \cdot \text{s} \]

\[ h_2 - h_1 = \Delta h = 25 + 13 - 5 - 10 = 1 + 33 \text{ m} = \Delta h \]

\[ V_{\text{average}} = \frac{\dot{V}}{\pi \frac{D^2}{4}} = \frac{5}{\pi \times \frac{2^2}{4}} = 1.592 \text{ m/s} = V_{\text{average}} \]

6.1)

\[ [\text{Re}]_D = \frac{V_{\text{average}} D}{\mu} = \frac{10^3 \times 1.592 \times 2}{10^{-3}} \]

\[ [\text{Re}]_D = 3.184 \times 10^6 \]

\[ > 4000 \]

\[ \rightarrow \text{flow is turbulent} \]
6.2) \[
\frac{E}{D} = \frac{0.1 \times 10^{-3}}{2} = 5 \times 10^{-5} = \frac{E}{D}
\]

\[ [Re]_d = 3.184 \times 10^6 \]

\textit{ Moody diagram (Fig. 2)}

\[ f = 0.0115 \]

\[ |\Delta P|_{\text{friction}} = f \frac{L}{D} \frac{1}{2} \rho \frac{V^2}{\text{average}} \]

\[ = 0.0115 \times \frac{8 \times 10^3}{2} \times 0.3 \times 10^3 \times 1.592^2 \]

\[ |\Delta P|_{\text{friction}} = 5.823 \times 10^4 \text{ Pa} = 0.583 \text{ bar} \]
6.3) \( \Delta p_{\text{height}} = \rho g \Delta h = 10^3 \times 9,81 \times 33 \)

\[
\Delta p_{\text{height}} = 32,373 \times 10^4 \text{ Pa} = 3,237 \text{ bar}
\]

\[
W_{\text{turbine}} = \dot{V} \left( |\Delta p|_{\text{height}} - |\Delta p|_{\text{friction}} \right)
\]

\[
= 5 \times \left( 32,373 - 5,889 \right) \times 10^4
\]

\[
W_{\text{turbine}} = 1,3272 \times 10^6 \text{ W} = 1,33 \text{ MW}
\]

6.4) at constant \( \dot{V} \),
when \( D \neq V \text{ average } \dot{V} \) and so (despite the moderate increase in \( \Delta p \), friction will decrease).

\( \rightarrow \) So the power will increase.

6.5) Yes because the \( |\Delta p|_{\text{total}} \) is not affected by the position of the turbine. However, there is increased risk of cavitation because of the lower absolute pressure at exit of turbine.