Fluid Dynamics
Chapter 7 – Pipe flows

last edited September 3, 2020
by Olivier Cleynen — https://fluidmech.ninja/

These notes are based on textbooks by White [22], Çengel & al.[25], Munson & al.[29], and de Nevers [17].

7.1 Motivation

In this chapter we focus on fluid flow in pipes. This topic allows us to explore several important phenomena with only very modest mathematical complexity. In particular, we are trying to answer two questions:

1. What does it take to describe fluid flow in ducts?

2. How can we quantify pressure changes in pipes and the power necessary to overcome them?

7.2 Frictionless flow in pipes

We begin with the simplest possible ducted flow case: a purely hypothetical fully-inviscid, incompressible, steady fluid flow in a one-dimensional pipe. Since there are no shear forces, the velocity profile across the duct remains uniform (flat) all along the flow, as shown in figure 7.1.

If the cross-sectional area A is changed, then the principle of mass conservation (eqs. 1/24, 3/6) is enough to allow us to compute the change in velocity $u = V$:

$$\rho V_1 A_1 = \rho V_2 A_2 \tag{7/1}$$

in steady pipe flow.

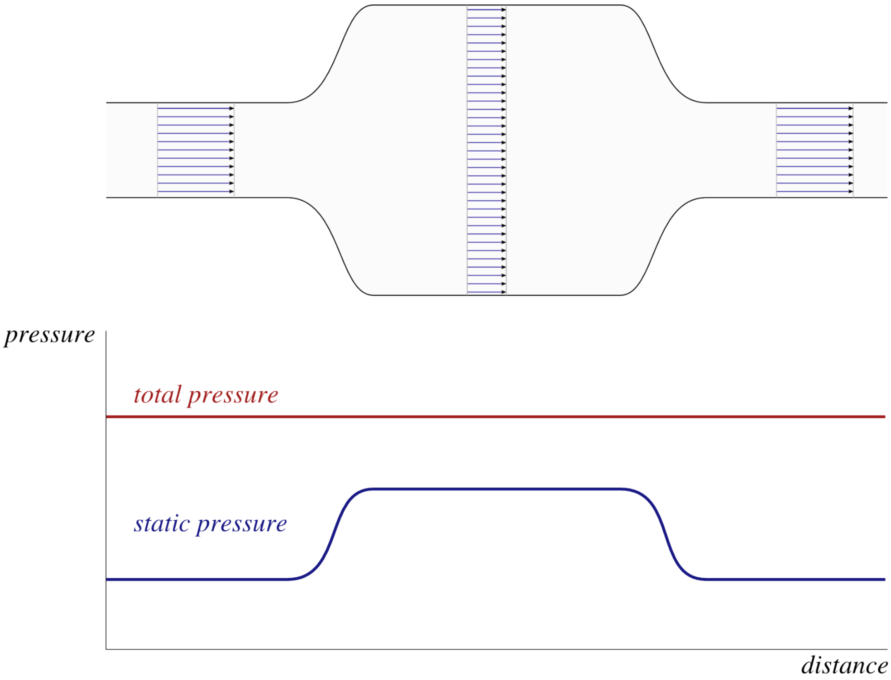


Figure 7.1: Inviscid fluid flow in a one-dimensional duct. In this purely hypothetical case, the velocity distribution is uniform across a cross-section of the duct. The average velocity and pressure change with cross-section area, but the total pressure $p_0 = p + \frac{1}{2}\rho V_{\text{av.}}^2$ remains constant.

Abstruse Goose #517: Daniel Bernoulli was part of a big family[W]

This information, in turn allows us to compute the pressure change between two sections of different areas by using the principle of energy conservation. We notice that the flow is so simple that the five conditions associated with the use of the Bernoulli equation (see §2.6 p. 41) are fulfilled: the flow is steady, incompressible, one-dimensional, has known trajectory, and does not feature friction or energy transfer. A simple application of eq. 2/20 p. 42 between any two points 1 and 2 gives us:

$$p_2 - p_1 = -\frac{1}{2}\rho\left[V_2^2 - V_1^2\right] - \rho g(z_2 - z_1) \tag{7/2}$$

in steady, incompressible, inviscid pipe flow without heat or work transfer.

Thus, in this kind of simple flow, pressure increases everywhere the velocity decreases, and vice-versa.

Another way of writing this equation is by stating that at constant altitude, the *total* or *dynamic pressure* $p_{\text{total}} \equiv p_0 \equiv p + \frac{1}{2}\rho V^2$ remains constant:

$$p_0 = \text{cst.} \tag{7/3}$$

at constant altitude, in laminar inviscid straight pipe flow.

Inviscid flows are nice, but real flows are more interesting. Real flows feature losses due to viscosity, and with viscous effects, one key assumption of the Bernoulli equation breaks down. An additional term will appear in the Bernoulli equation, as we have seen in chapter 2 with equation 2/21 p. 43. What does this extra term Δp_{loss} depend on, and how can we quantify it? This is what the rest of the chapter is about.

7.3 Parameters to quantify losses in pipes

Hydraulics is the oldest branch of fluid dynamics, and much of the notation used to describe pressure losses predates modern applications. The most widely-used parameters for quantifying losses due to friction in a duct are the following:

The pressure loss is the most intuitive way of quantifying the net effect of friction in pipes. Engineers and physicists usually quantify Δp_{loss} as a negative number (i.e. $\Delta p_{\text{loss}} \equiv p_2 - p_1$ with $p_2 < p_1$). However, in hydraulics, the historical precedent is to quantify pressure loss with a positive number. To make clear this convention, we will always refer to pressure loss as $|\Delta p_{\text{loss}}|$.

The elevation loss which we note $|\Delta l|$ (in the literature, often noted Δh), is defined as

$$|\Delta l| \equiv \frac{|\Delta p_{\text{loss}}|}{\rho g} \tag{7/4}$$

It represents the hydrostatic height loss (with a positive number) caused by the fluid flow in the duct, and is measured in meters. The reference density ρ in this definition is taken as the density of the fluid, the density of water, or the density of mercury, depending on cases. We do not use this definition in this course.

The Darcy friction factor noted f is defined as

$$f \equiv \frac{|\Delta p_{\text{loss}}|}{\frac{L}{D}\frac{1}{2}\rho V_{\text{av.}}^2} \tag{7/5}$$

where $V_{\text{av.}}$ is the average flow velocity in the pipe.

In pipe flows of relevance to the engineer, f has values between $5 \cdot 10^{-5}$ and $5 \cdot 10^{-2}$. Those values can be calculated or read from experimental data, as explained further down.

The loss coefficient noted K_L is defined as

$$K_L \equiv \frac{|\Delta p_{\text{loss}}|}{\frac{1}{2}\rho V_{\text{av.}}^2} \tag{7/6}$$

Components found in pipe networks, such as bends, filter screens, valves, or junction screens, all result in losses that remain roughly proportional to the square of the average velocity. Typically, K_L values range between 0,3 (smooth bend) and 2 (partially-closed valve).

7.4 Laminar flow in pipes

7.4.1 Laminar flow between plates

Before we study fluid flow in a *circular* pipe, let us begin with a simpler case: flow between two parallel plates. This is a good place to start, because we can work with Cartesian coordinates, and focus on two dimensions only.

Let us first try a qualitative description of the flow, as displayed in figure 7.2. Because of the no-slip condition at the walls, the velocity distribution within any cross-flow section cannot be uniform. Shear occurs, which translates into a pressure decrease along the flow. The faster the flow, and the higher the gradient of velocity. Thus, shear within the flow, and the resulting pressure loss, both increase when the cross-sectional area is decreased.

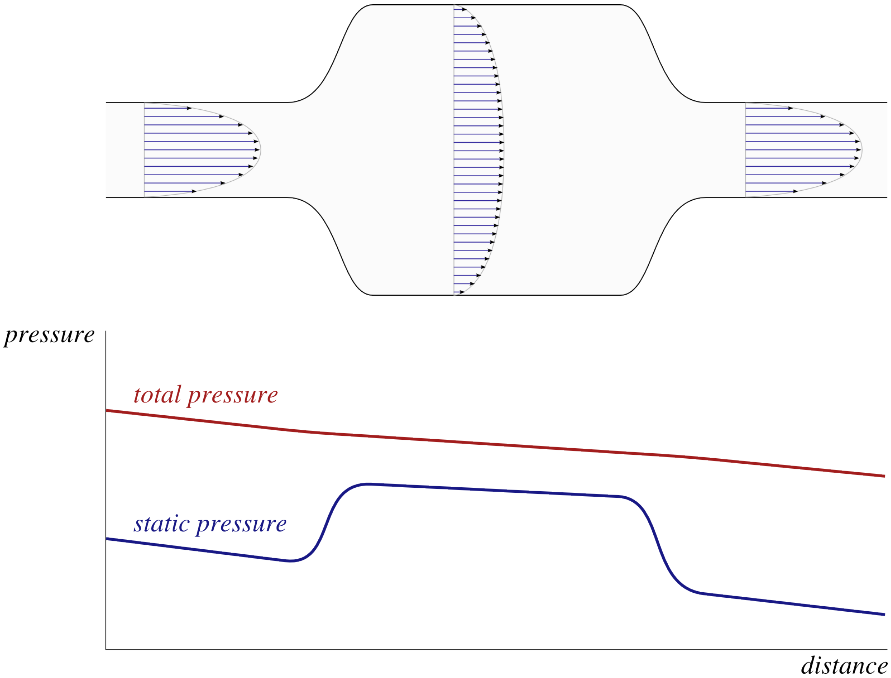


Figure 7.2: Viscous laminar fluid flow in a one-dimensional pipe. This time, the no-slip condition at the wall creates a viscosity gradient across the duct cross-section. This in turn translates into pressure loss. Sudden duct geometry changes such as represented here would also disturb the flow further, but the effect was neglected here.

Figure CC-0 *Olivier Cleynen*

How can we now describe *quantitatively* the velocity profile and the pressure loss? We need to clearly sketch the flow we are interested in, which we do in figure 5.10.

We also need a powerful, extensive mathematical tool to describe the flow: we turn to the Navier-Stokes equation which we derived in the previous chapter as eq. 6/42 p. 125:

$$\rho \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \rho \vec{g} - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{V} \tag{7/7}$$

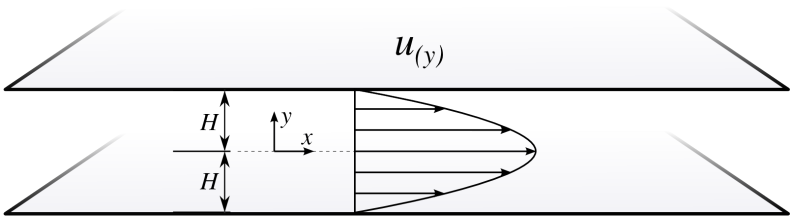


Figure 7.3: Two-dimensional laminar flow between two plates, also called *Poiseuille flow*. We already studied this flow case in fig. 5.10 p. 106; this time, we wish to derive an expression for the velocity distribution.

Figure CC-0 *Olivier Cleynen*

Since we are applying this tool to the simple case of fully-developed, two-dimensional incompressible fluid flow between two parallel plates (fig. 7.3), we need only two Cartesian coordinates, so that the vector equation translates to:

$$\rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = \rho g_x - \frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial y)^2} \right] \tag{7/8}$$

$$\rho \left[\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right] = \rho g_y - \frac{\partial p}{\partial y} + \mu \left[\frac{\partial^2 v}{(\partial x)^2} + \frac{\partial^2 v}{(\partial y)^2} \right] \tag{7/9}$$

In this particular flow, we have restricted ourselves to a fully-steady ($\mathrm{d}/\,\mathrm{d}t = 0$), horizontal ($g = g_y$), one-directional flow ($v = 0$). When the flow is fully developed, $\partial u/\partial x = 0$ and $\partial^2 u/(\partial x)^2 = 0$, and the system above shrinks down to:

$$0 = -\frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 u}{(\partial y)^2} \right] \tag{7/10}$$

$$0 = \rho g - \frac{\partial p}{\partial y} \tag{7/11}$$

We only have to integrate equation 7/10 twice with respect to y to come to the velocity profile across two plates separated by a height $2H$:

$$u = \frac{1}{2\mu} \left(\frac{\partial p}{\partial x} \right) (y^2 - H^2) \tag{7/12}$$

Now, the longitudinal pressure gradient $\partial p/\partial x$ can be evaluated by working out the volume flow rate $\dot{\mathcal{V}}$ for any given width Z with one further integration of equation 7/12:

$$\frac{\dot{\mathcal{V}}}{Z} = \frac{2}{Z} \int_0^H uZ \,\mathrm{d}y = -\frac{2H^3}{3\mu} \left(\frac{\partial p}{\partial x} \right)$$

$$\frac{\partial p}{\partial x} = -\frac{3}{2} \frac{\mu}{ZH^3} \dot{\mathcal{V}} \tag{7/13}$$

In this section, the overall process is more important than the result: by starting with the Navier-Stokes equations, and adding known constraints that describe the flow of interest, we can predict analytically all of the characteristics of a laminar flow.

7.4.2 Laminar flow in pipes

We now turn to studying flow in *cylindrical* pipes, which are widely used; first considering laminar flow, and then expanding to turbulent flow.

The process is identical to above, only applied to cylindrical instead of Cartesian coordinates. We focus on the fully-developed laminar flow of a fluid in a cylindrical pipe without gravity (fig. 7.4).

For this flow, we wish to work out the velocity profile and calculate the pressure loss related to the flow.

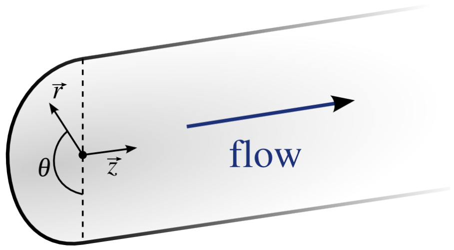


Figure 7.4: A cylindrical coordinate system to study laminar flow in a cylindrical duct.

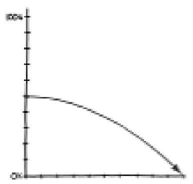
Figure CC-0 *Olivier Cleynen*

We once again start from the Navier-Stokes vector equation, choosing this time to develop it using *cylindrical* coordinates:

$$\rho\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right] = \rho g_r - \frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{(\partial\theta)^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial^2 v_r}{(\partial z)^2}\right] \tag{7/14}$$

$$\rho\left[\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right] = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{(\partial\theta)^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} + \frac{\partial^2 v_\theta}{(\partial z)^2}\right] \tag{7/15}$$

$$\rho\left[\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right] = \rho g_z - \frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{(\partial\theta)^2} + \frac{\partial^2 v_z}{(\partial z)^2}\right] \tag{7/16}$$



XKCD #1230: polar coordinates
by Randall Munroe (CC-BY-NC)
https://xkcd.com/1230

This mathematical arsenal does not frighten us, for the simplicity of the flow we are studying allows us to bring in numerous simplifications. First, we have $g = 0$. Second, we have $v_r = 0$ and $v_\theta = 0$ everywhere. Thus, by continuity, $\partial v_z/\partial z = 0$.
Furthermore, since our flow is symmetrical, v_z is independent from θ. With these two conditions, the above system shrinks down to:

$$0 = 0 \tag{7/17}$$

$$0 = 0 \tag{7/18}$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] \tag{7/19}$$

Now, with equation 7/19, we work towards obtaining an expression for v_z by integrating twice our expression for $\partial v_z/\partial r$:

$$\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{r}{\mu}\frac{\partial p}{\partial z}$$

$$\left(r\frac{\partial v_z}{\partial r}\right) = \frac{r^2}{2\,\mu}\left(\frac{\partial p}{\partial z}\right) + k_1$$

$$v_z = \frac{r^2}{4\,\mu}\left(\frac{\partial p}{\partial z}\right) + k_1 \ln r + k_2 \tag{7/20}$$

We have to use boundary conditions so as to unburden ourselves from integration constants k_1 and k_2.
By setting $v_{z@r=0}$ as finite, we deduce that $k_1 = 0$ (because $\ln(0) \longrightarrow -\infty$).
By setting $v_{z@r=R} = 0$ (no-slip condition), we obtain $k_2 = -\frac{R^2}{4}\frac{\partial p}{\mu}\frac{\partial p}{\partial z}$.

This simplifies eq. (7/20) and brings us to our objective, an extensive expression for the velocity profile across a pipe of radius R when the flow is laminar:

$$v_z = u_{(r)} = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(R^2 - r^2) \tag{7/21}$$

This equation is parabolic (fig. 7.5). It tells us that in a pipe of given length L and radius R, a given velocity profile will be achieved which is a function only of the ratio $\Delta p/\mu$.

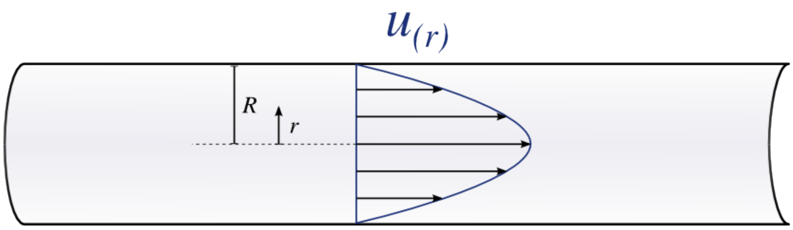


Figure 7.5: The velocity profile across a cylindrical pipe featuring laminar viscous flow.

We can also express the pressure gradient in the pipe as a function of the volume flow rate. This is done through integration of velocity with respect to density and cross-section area. We obtain:

$$\dot{m} = -\frac{\pi \rho D^4}{128\,\mu}\left(\frac{\partial p}{\partial z}\right)$$
$$\Delta p_{\text{loss}} = -\frac{128}{\pi}\frac{\mu L \dot{m}}{\rho D^4} \tag{7/22}$$

This equation is interesting in several respects. For a given pipe length L and pressure drop Δp_{loss}, the volume flow $\dot{\mathcal{V}}$ increases with the power 4 of the diameter D. In other words, the volume flow is multiplied by 16 every time the diameter is doubled.

We also notice that the pipe wall roughness does not appear in equation 7/22. In a laminar flow, increasing the pipe roughness has no effect on the velocity distribution in the pipe.

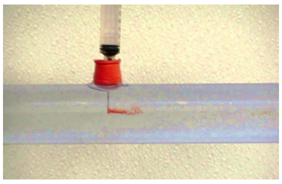
Out of curiosity, we may translate the result in equation 7/22 into a friction factor equation, with the help of definition 7/5, obtaining

$$f_{\text{laminar cylinder flow}} = \frac{32 V_{\text{av.}} \mu L}{\frac{L}{D}\frac{1}{2}\rho V_{\text{av.}}^2 D^2} = 64\frac{\mu}{\rho V_{\text{av.}}\, D} = \frac{64}{[\text{Re}]_D} \tag{7/23}$$

in which we inserted the average velocity $V_{\text{av.}} = \frac{\dot{\mathcal{V}}}{\pi R^2} = -\frac{\Delta p\, D^2}{32 \mu L}$.

7.5 Turbulent flow in pipes

7.5.1 When is a pipe flow turbulent?



Video: very basic visualization of laminar and turbulent flow regimes in a transparent pipe
by Engineering Fundamentals (STYL)
https://youtu.be/56AyTIhNQBo

It has long been observed that pipe flow can have different *regimes*. In some conditions, the flow is unable to remain laminar (one-directional, fully-steady); it becomes *turbulent*. Although the flow is steady when it is averaged over short time period (e.g. a few seconds), it is subject to constant, small-scale, chaotic and spontaneous velocity field changes in all directions.

In 1883, Osborne Reynolds published the results of a meticulous investigation into the conditions in which the flow is able, or not, to remain laminar (figs. 7.6 and 7.7). He showed that they could be predicted using a single non-dimensional parameter, later named *Reynolds number*, which, as we have seen already with eq. 1/28 p. 23, is expressed as:

$$[\mathrm{Re}] \equiv \frac{\rho\, V\, L}{\mu} \tag{7/24}$$

In the case of pipe flow, the representative length L is conventionally set to the pipe diameter D and the velocity to the cross-section average velocity:

$$[\mathrm{Re}]_D \equiv \frac{\rho\, V_{\text{av.}}\, D}{\mu} \tag{7/25}$$

where $V_{\text{av.}}$ is the average velocity in the pipe ($\mathrm{m\,s^{-1}}$),
and D is the pipe diameter (m).

The occurrence of turbulence is very well documented. The following values are widely accepted:

- Pipe flow is laminar for $[\mathrm{Re}]_D \lesssim 2\,300$;

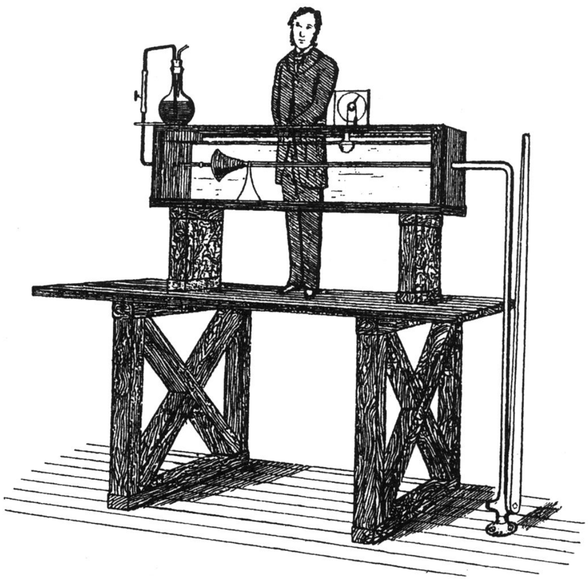


Figure 7.6: Illustration published by Reynolds in 1883 showing the installation he set up to investigate the onset of turbulence. Water flows from a transparent rectangular tank down into a transparent drain pipe, to the right of the picture. Colored die is injected at the center of the pipe inlet, allowing for the visualization of the flow regime.

Image by Osborne Reynolds (1883, public domain)

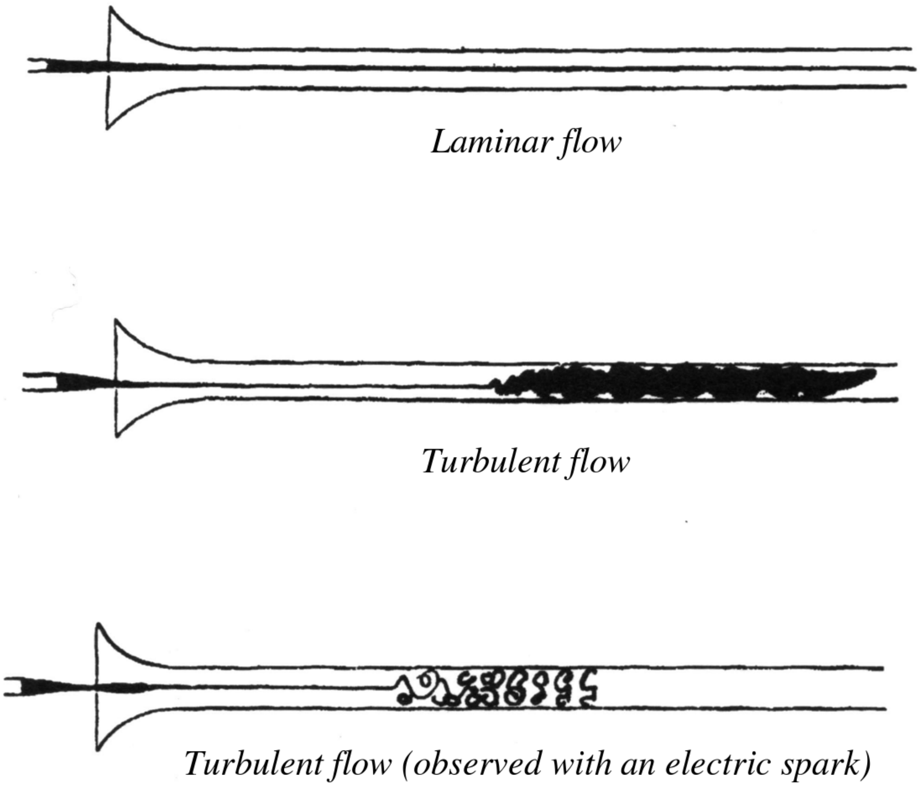


Figure 7.7: Illustration published by Reynolds in 1883 showing two different flow regimes observed in the installation from fig. 7.6.

Image by Osborne Reynolds (1883, public domain)

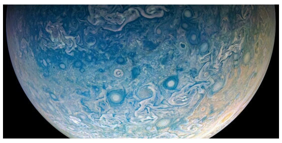
- Pipe flow is turbulent for $[\text{Re}]_D \gtrsim 4\,000$.

The significance of the Reynolds number extends far beyond pipe flow; we shall explore this in chapter 8 (*Engineering models*).

7.5.2 Characteristics of turbulent flow

This topic is well covered in Tennekes & Lumley [5]

Turbulence is a complex topic which is still not fully described analytically today. Although it may display steadiness when time-averaged, a turbulent flow is highly three-dimensional, unsteady, and chaotic in the sense that the description of its velocity field is carried out with statistical, instead of analytic, methods.



Video: turbulence for those who don't have time to read chapter 9
by Y:Veritasium (STYL)
https://youtu.be/5zI9sG3pjVU

In the scope of our study of fluid dynamics, the most important characteristics associated with turbulence are the following:

- A strong increase in mass and energy transfer within the flow. Slow and rapid fluid particles have much more interaction (especially momentum transfer) than within laminar flow;
- A strong increase in losses due to friction (typically by a factor 2). The increase in momentum exchange within the flow creates strong dissipation through viscous effects, and thus transfer (as heat) of macroscopic forms of energy (kinetic and pressure energy) into microscopic forms (internal energy, translating as temperature);

- Internal flow movements appear to be chaotic (though not merely *random*, as would be white noise), and we do not have mathematical tools to describe them analytically.

Consequently, solving a turbulent flow requires taking account of flow in all three dimensions even for one-directional flow! We will come back to this topic in chapter 9 (*Dealing with turbulence*).

7.5.3 Velocity profile in turbulent pipe flow

Video: highly-turbulent flow exiting the flood discharge ducts of the Tarbela dam in northeastern Pakistan

by Y:Beauty Of Pakistan (STYL)
https://youtu.be/13tBWzKajqw

In order to deal with the vastly-increased complexity of turbulent flow in pipes, we split each velocity component v_i in two parts, a time-averaged component $\overline{v_i}$ and an instantaneous fluctuation v_i':

$$v_r = \overline{v_r} + v_r'$$
$$v_\theta = \overline{v_\theta} + v_\theta'$$
$$v_z = \overline{v_z} + v_z'$$

In our case, $\overline{v_r}$ and $\overline{v_\theta}$ are both zero, but the fluctuations v_r' and v_θ' are not, and will cause v_z to differ from the laminar flow case. The extent of turbulence is often measured with the concept of *turbulence intensity* I:

$$I \equiv \frac{\left[\overline{v_i'^2}\right]^{\frac{1}{2}}}{\overline{v_i}} \tag{7/26}$$

Regrettably, we have not found a general analytical solution to turbulent pipe flow — note that if we did, it would likely exhibit complexity in proportion to that of such flows. A widely-accepted average velocity profile constructed from experimental observations is:

$$\overline{u}_{(r)} = \overline{v_z} = \overline{v_{z\ \text{max}}} \left(1 - \frac{r}{R}\right)^{\frac{1}{7}} \tag{7/27}$$

While it closely and neatly matches experimental observations, this model is nowhere as potent as an analytical one and must be seen only as an approximation. For example, it does not allow us to predict internal energy dissipation rates (because it describes only time-averaged velocity), or even wall shear stress (because it yields $(\partial\overline{u}/\partial r)_{r=R} = \infty$, which is not physical).

The following points summarize the most important characteristics of turbulent velocity profiles:

- They continuously fluctuate in time and we have no means to predict them extensively;
- They are much “flatter” than laminar profiles (fig. 7.8);
- They depend on the wall roughness;
- They result in shear and dissipation rates that are markedly higher than laminar profiles.

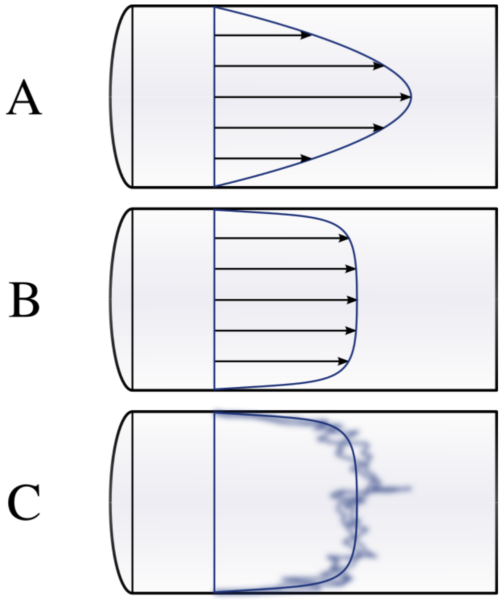


Figure 7.8: Velocity profiles for laminar (A), and turbulent (B and C) flows in a cylindrical pipe. B represents the time-averaged velocity distribution, while C shows several arbitrary instantaneous distributions (blurred) as well as their average in time. Turbulent flow in a pipe also features velocities in the radial and angular directions, which are not shown here.

Figure CC-0 *Olivier Cleynen*

7.5.4 Pressure losses in turbulent pipe flow

Losses caused by turbulent flow depend on the wall roughness ϵ and on the diameter-based Reynolds number $[\text{Re}]_D$.

For lack of an analytical solution, we are not able to predict the value of the friction factor f anymore. Several empirical models can be built to obtain f, the most important of which is known as the *Colebrook equation* expressed as:

$$\frac{1}{\sqrt{f}} = -2 \log \left(\frac{1}{3{,}7} \frac{\epsilon}{D} + \frac{2{,}51}{[\text{Re}]_D \sqrt{f}} \right) \tag{7/28}$$

The structure of this equation makes it inconvenient to solve for f. To circumvent this difficulty, equation 7/28 can be solved graphically on the *Moody diagram*, fig. 7.9. This classic document allows us to obtain numerical values for f (and thus predict the pressure losses Δp_{loss}) if we know the diameter-based Reynolds number $[\text{Re}]_D$ and the relative roughness ϵ/D.

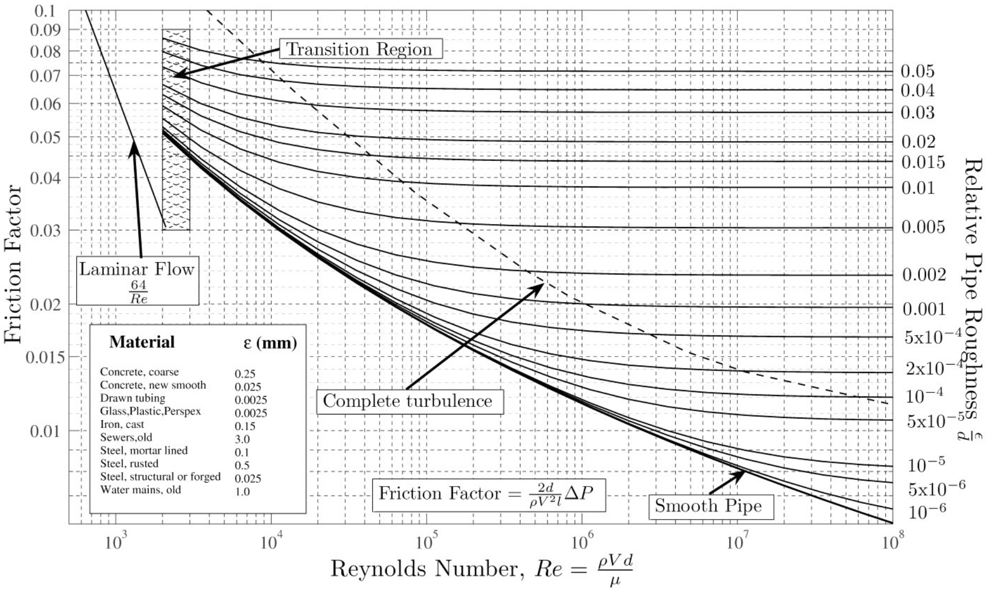


Figure 7.9: A Moody diagram, which presents values for f measured experimentally, as a function of the diameter-based Reynolds number $[\mathrm{Re}]_D$, for different values of the relative roughness ϵ/D. This figure is reproduced with a larger scale as figure 7.11 p. 152.

Diagram CC-BY-SA *S Beck and R Collins, University of Sheffield*

7.6 Engineer's guide to pipe flows

7.6.1 Summary so far

The world of pipe flows is a good playground for us to experiment with practical fluid mechanics. The main lessons learned in this chapter are as follows:

- From an engineering point of view, pipe flows are one-dimensional. We push a volume flow of fluid in on one side and expect to receive it on the other, perhaps with different properties. The integral "cost" of doing so is quantified with a single value, Δp_{loss}.
- When the flow is laminar, we can solve for the flow in the pipe. For this, we write the Navier-Stokes equations and apply known boundary conditions. We obtain the velocity *everywhere* in the pipe, which gives us a wealth of information about the flow, including the Δp_{loss}.
- When the flow is turbulent, this analysis method does not work any-more. Even though the conditions are in principle simple, the flow is mesmerizingly complex. We resort to building models based on *time-averaged measurement data.* Those models work for a wide number of pipes and conditions because they relate parameters which are *non-dimensionalized.* We obtain the desired Δp_{loss} with a single diagram reading and a simple algebraic manipulation.

These broad trends apply to many other sub-areas of fluid mechanics.

7.6.2 Choosing laminar or turbulent flow

When designing a pipe system, an engineer may have the opportunity to create laminar or turbulent flow. Inserting an expression for mass flow

$\dot{m} = \rho V_{\text{av.}}(\pi/4)D^2$ into the definition of the Reynolds number, we obtain, for pipe flow:

$$[\text{Re}] = \frac{\dot{m}}{D}\frac{4}{\pi\mu} \tag{7/29}$$

This equation is telling: for a given fluid (μ) and a given mass flow ($\dot{m}$), the only way to ensure that the flow is laminar (with low Reynolds number) is to increase the diameter D. Doing so also increases installation costs; therefore, there is a balance to strike between initial installation costs (increasing with diameter) and operating costs due to pressure losses (which decrease as the diameter increases). In practice, except for cases where very small mass flows and velocities are involved (e.g. medical fluid flows), most piping installations feature turbulent flow.

7.6.3 Pressure losses in laminar flow

The pressure losses in laminar flow are summed up with equation 7/22 p. 141, which we repeat here with the most important terms positioned first:

$$\Delta p_{\text{loss}} = -L\frac{\dot{m}}{D^4}\frac{128\mu}{\pi\rho} \tag{7/30}$$

for laminar pipe flow.

Again, it is visible here that in laminar pipe flow, losses per unit pipe length increase linearly with mass flow $\dot{m}$, and with the power −4 (!) of the diameter D.

7.6.4 Pressure losses in turbulent flow

The picture for losses in turbulent flow is harder to draw. Re-arranging the definition 7/5 p. 137 to include the mass flow, we obtain:

$$\Delta p_{\text{loss}} = -L\frac{\dot{m}^2}{D^5}\,f\,\frac{8}{\pi^2\rho} \tag{7/31}$$

for turbulent pipe flow.

The factor f in this equation generally varies according to the Reynolds number and to the roughness of the pipe, as described in the Moody diagram. At very turbulent regimes (in the upper right area of the diagram), f becomes independent of $[\text{Re}]_D$ and proportional to the relative roughness, so that we can write:

$$\Delta p_{\text{loss}} = -L\frac{\dot{m}^2}{D^5}\left(c_1 + c_2\frac{\epsilon}{D}\right)\frac{8}{\pi^2\rho} \tag{7/32}$$

for turbulent pipe flow in very turbulent regimes,
where c_1 and c_2 are approximately constant.

Based on this equation, we can see that pressure losses in highly-turbulent pipe flow increase approximately with the square of mass flow $\dot{m}$ and the power −5 of the diameter D. In between this regime and the laminar regime, a variety of intermediary states are quantified using the Moody diagram.

7.6.5 Calculating pumping and turbining power

The tools above are all we need to calculate, given the geometry of an installation, how much pumping or turbining power is involved in moving a given mass flow of liquid though a pipe system. The steps are as follows:

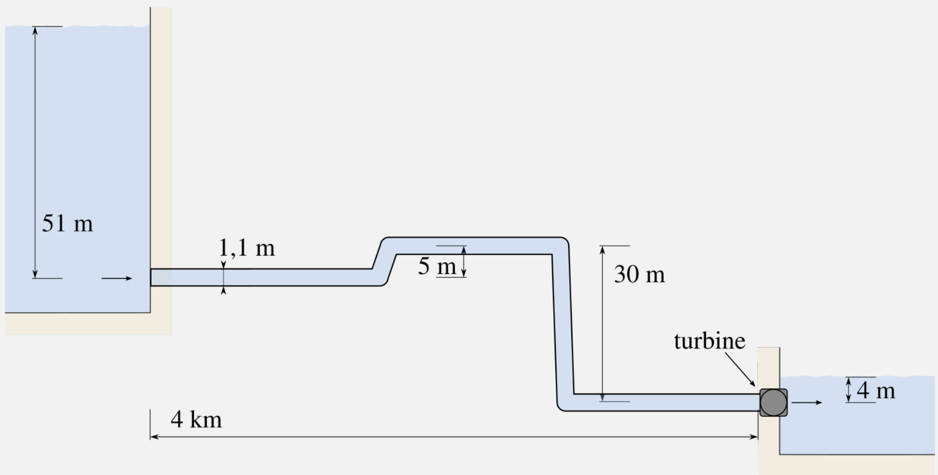
1. Calculate the hydrostatic pressure drop across the device.
   This is done by imagining that there is no flow, and calculating the static pressure which would then exert on each side of the device, using equation 4/15 p. 80.

2. Calculate the pressure losses due to friction.
   This is done by calculating the Reynolds number, using the Moody diagram to read the corresponding friction factor f, and calculating the corresponding Δp_{loss} using the definition 7/5. Care must be taken with the sign of Δp_{loss}, which is *always negative* by definition, but very often expressed as a positive number in the literature.
   Pressure losses induced by bends, junctions and obstacles are likewise calculated using their K_L values and the definition 7/6.

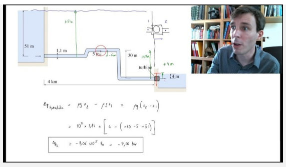
3. Summing up the pressure differences.
   The complete pressure difference across the device is $\Delta p_{\text{device}} = -\Delta p_{\text{losses}} + \Delta p_{\text{hydrostatic}}$. Pressure losses due to friction are always negative, and hydrostatic pressure differences may have either sign.
   The power is recovered using equation 1/22 p. 19, $\dot{P}_{\text{device}} = \Delta p_{\text{device}}\ \dot{m}/\rho$.
   If this power is negative, the liquid is losing energy, and the device is acting as a turbine. If the power is positive, the liquid is gaining energy, and the device is acting as a pump.

7.7 Solved problems

Hydrostatic pressure on a turbine



A turbine is installed as shown above. What is the hydrostatic pressure difference available to the turbine?



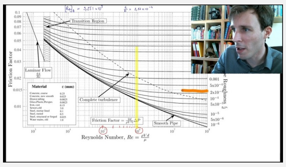
See this solution worked out step by step on YouTube
https://youtu.be/3lyKby0fS-8 (CC-BY Olivier Cleynen)

Pressure loss in a pipe

In the piping installation from the previous example, water at 20 °C is flowing with a volume flow of $800\,\mathrm{L\,s^{-1}}$.

The pipe has roughness $\epsilon = 0{,}25$ mm and a diameter $D = 1{,}1$ m. The bends each induce a loss coefficient $K_L = 0{,}75$.

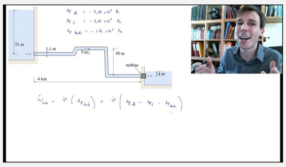
What is the pressure drop due to friction losses in the pipe?



See this solution worked out step by step on YouTube
https://youtu.be/Tp6a_50uqUc (CC-BY Olivier Cleynen)

Turbining power

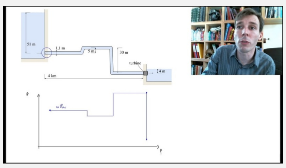
In the piping installation from the previous examples, what is the power made available to the turbine?



See this solution worked out step by step on YouTube
https://youtu.be/BOew8INdQ54 (CC-BY Olivier Cleynen)

Pressure distribution in a pipe

In the piping installation from the previous examples, what is the pressure distribution along the pipe?



See this solution worked out step by step on YouTube
https://youtu.be/q1xOmWaYZLg (CC-BY Olivier Cleynen)

Problem sheet 7: Pipe flows

last edited June 30, 2021
by Olivier Cleynen — https://fluidmech.ninja/

Except otherwise indicated, assume that:
The atmosphere has $p_{\text{atm.}} = 1\,\text{bar}$; $\rho_{\text{atm.}} = 1{,}225\,\text{kg}\,\text{m}^{-3}$; $T_{\text{atm.}} = 11{,}3\,°\text{C}$; $\mu_{\text{atm.}} = 1{,}5 \cdot 10^{-5}\,\text{Pa}\,\text{s}$
Air behaves as a perfect gas: R_{air}=287 J kg^{-1} K^{-1}; γ_{air}=1,4; $c_{p\,\text{air}}$=1 005 J kg^{-1} K^{-1}; $c_{v\,\text{air}}$=718 J kg^{-1} K^{-1}
Liquid water is incompressible: $\rho_{\text{water}} = 1\,000\,\text{kg}\,\text{m}^{-3}$, $c_{p\,\text{water}} = 4\,180\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}$

In cylindrical pipe flow, we assume the flow is always laminar for $[\text{Re}]_D \lesssim 2\,300$, and always turbulent for $[\text{Re}]_D \gtrsim 4\,000$. The Darcy friction factor f is defined as:

$$f \equiv \frac{|\Delta p_{\text{loss}}|}{\frac{L}{D}\frac{1}{2}\rho V_{\text{av.}}^2} \qquad (7/5)$$

The loss coefficient K_L is defined as:

$$K_L \equiv \frac{|\Delta p_{\text{loss}}|}{\frac{1}{2}\rho V_{\text{av.}}^2} \qquad (7/6)$$

Viscosities of various fluids are given in fig. 7.10. Pressure losses in cylindrical pipes can be calculated with the help of the Moody diagram presented in fig. 7.11 p. 152.

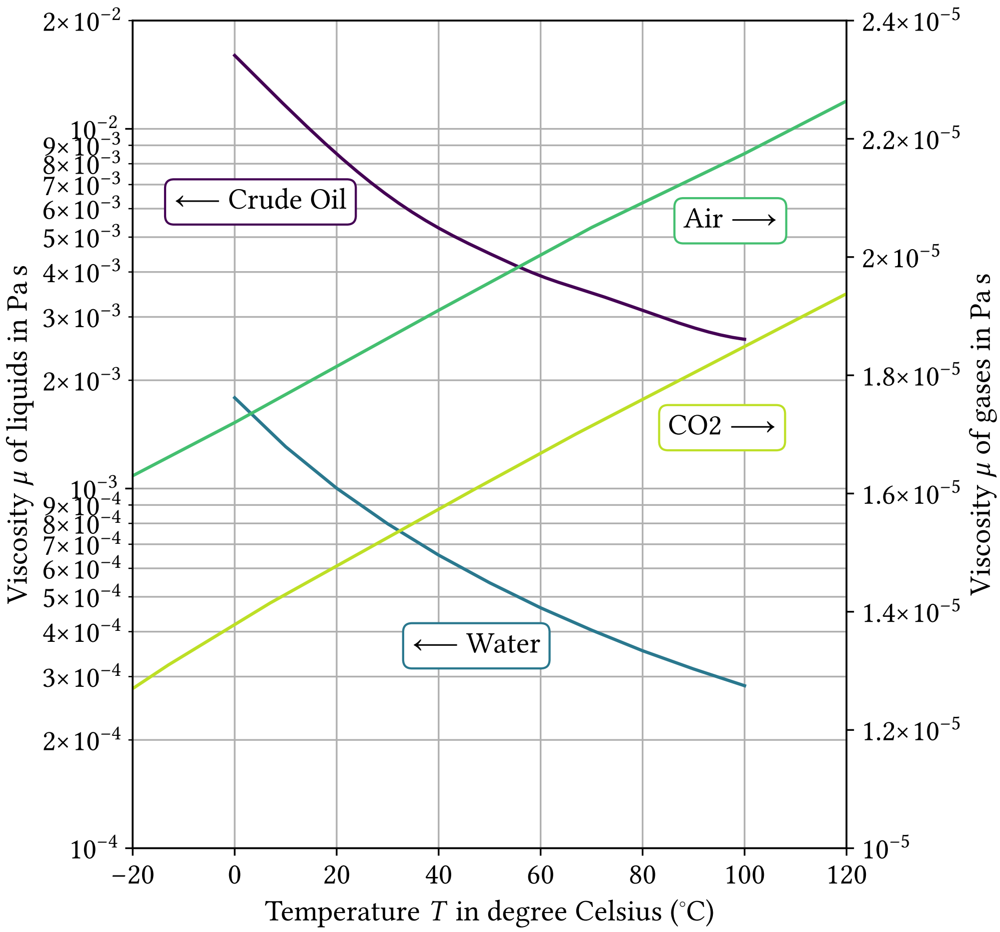


Figure 7.10: The viscosity of four fluids (crude oil, water, air, and C02) as a function of temperature. The scale for liquids is logarithmic and displayed on the left; the scale for gases is linear and displayed on the right.

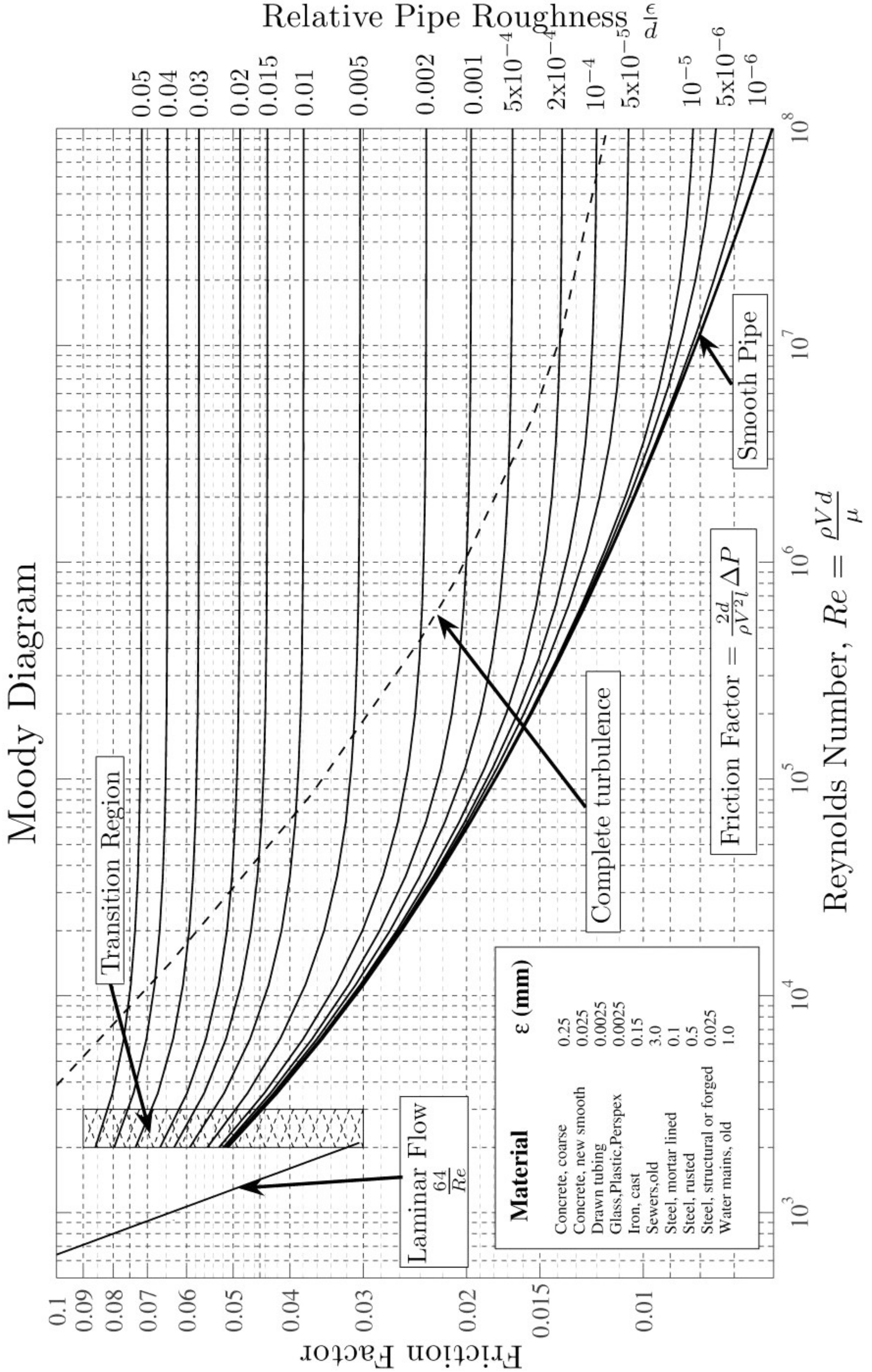


Figure 7.11: A Moody diagram, which presents values for f measured experimentally, as a function of the diameter-based Reynolds number $[\mathrm{Re}]_D$, for different relative roughness values.

7.1 Reading quiz

Once you are done with reading the content of this chapter, you can go take the associated quiz at https://elearning.ovgu.de/course/view.php?id=7199

In the winter semester, quizzes are not graded.

7.2 Revision questions

non-examinable

The Moody diagram (fig. 7.11 p. 152) is simple to use, yet it takes practice to understand it fully… here are three questions to guide your exploration. They can perhaps be answered as you work through the other examples.

7.2.1. Why is there no zero on the diagram?

7.2.2. Why are the curves sloped downwards — should friction losses not instead *increase* with increasing Reynolds number?

7.2.3. Why can the pressure losses Δp_{losses} be calculated given the volume flow $\dot{\mathcal{V}}$, but not the other way around?

7.3 Air flow in a small pipe

Munson & al. [29] E8.5

A machine designed to assemble micro-components uses an air jet. This air is driven through a 10 cm-long cylindrical pipe with a 4 mm diameter, roughness 0,0025 mm, at an average speed of $50\,\mathrm{m\,s^{-1}}$.

The inlet air pressure and temperature are 1,2 bar and 60 °C; the viscosity of air is quantified in fig. 7.10 p. 151.

7.3.1. What is the pressure loss caused by the flow through the pipe?

7.3.2. What is the maximum average flow speed for which the flow would remain laminar?

7.3.3. What would this speed be if the pipe diameter was 4 cm instead of 4 mm?

7.4 Water piping

A long pipe is installed to carry water from one large reservoir to another (fig. 7.16). The total length of the pipe is 10 km, its diameter is 0,5 m, and its roughness is $\epsilon = 0{,}5\,\mathrm{mm}$. It must climb over a hill, so that the altitude changes along with distance.

The pump must be powerful enough to push $1\,\mathrm{m^3\,s^{-1}}$ of water at 20 °C.

Figure 7.10 p. 151 quantifies the viscosity of various fluids, and fig. 7.11 p. 152 quantifies losses in cylindrical pipes.

7.4.1. Will the flow in the water pipe be turbulent?

7.4.2. What is the pressure loss caused by the flow through the pipe?

7.4.3. What is the pumping power required to meet the design requirements?

7.4.4. What would be the power required for the same volume flow if the pipe diameter was doubled?

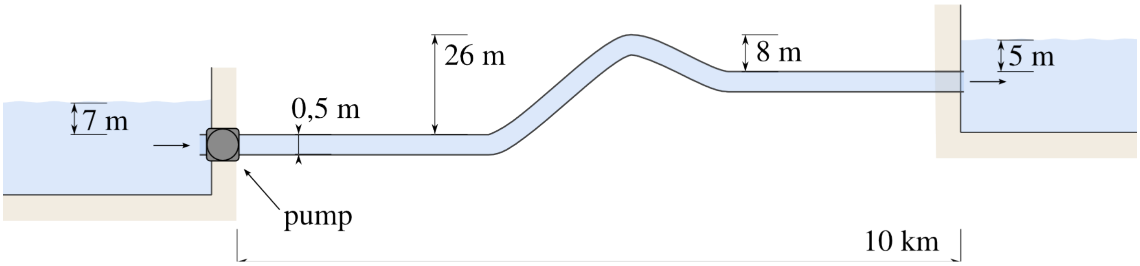


Figure 7.12: Layout of the water pipe. For clarity, the vertical scale is greatly exaggerated. The diameter of the pipe is also exaggerated.

7.5 Design of a water piping system

You start your career as a junior engineer in a company that designs piping and pumping systems.

Your first assignment is to choose the dimensions of a system which should carry $3\,\mathrm{m^3\,h^{-1}}$ of water ($10^{-3}\,\mathrm{Pa\,s}$) across a horizontal distance of $1\,\mathrm{km}$. Fresh from reading through chapter 7, you design the system to feature laminar flow only.

You begin with a revision of the relevant theory, starting, of course, from the Navier-Stokes equations and a simple diagram (fig. 7.13), obtaining the velocity distribution for laminar flow in a circular pipe:

$$v_z = u_{(r)} = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(R^2 - r^2) \tag{7/21}$$

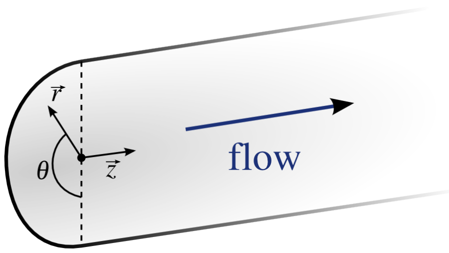


Figure 7.13: A cylindrical coordinate system to study laminar flow in a horizontal cylindrical duct.

7.5.1. What is the minimum pipe diameter for which the flow will remain laminar?

7.5.2. Starting from equation 7/21, show that the pressure loss per unit length in the pipe with laminar flow is expressed as a function of the volume flow $\dot{\mathcal{V}}$ and the diameter D as:

$$\Delta p_{\text{loss}} = -\frac{128}{\pi}\frac{\mu L \dot{m}}{\rho D^4} \tag{7/22}$$

7.5.3. With the diameter chosen above, what is the pressure loss in the pipe?

7.5.4. What is the pumping power required?

With those results in hand, you turn to your colleagues — but with a smile, they suggest you try a design with turbulent flow instead.

7.5.5. What would be the pressure loss if an 8 cm-diameter plastic pipe was used?

7.5.6. What would then be the pumping power required?

7.5.7. What is one advantage of using a pipe with smaller diameter? (briefly justify your answer, e.g. in 30 words or less)

7.6 Major oil pipeline

Strong from your experience working through problem 7.5, you join the team in charge of designing one very large oil pipeline system (fig. 7.14).

Your design must safely carry 700 thousand barrels of oil ($110\,000\ \mathrm{m^3}$) per day along a length of 1 200 km. The crude oil has density $900\ \mathrm{kg\,m^{-3}}$ and its viscosity is quantified in fig. 7.10 p. 151. The average temperature of the oil during the transit is 60 °C.

The landscape is flat for most of the journey, with a 200 km-wide mountain range in the middle that reaches 1 400 m altitude.

Your team selects a cylindrical, smooth steel duct with 1,22 m diameter, average roughness $\epsilon = 0{,}15\ \mathrm{mm}$. Because the pipeline passes through ecologically fragile areas, as well as a seismically-active region, you decide to never exceed 200 psi (13,8 bar) of gauge pressure in the pipeline. To prevent oil cavitation (a change of state with destructive consequences), you decide to never reach below 0,8 bar of absolute pressure in the pipeline.

7.6.1. How much time does the average oil particle need to travel across the line?

7.6.2. How much pumping power is required in total?

7.6.3. How far apart should the pumping stations be laid out in the flat sections of the pipeline?

7.6.4. How far apart should the pumping stations be laid out in the ascending section of the pipeline?

7.6.5. Propose a pumping station arrangement, and calculate the power required for each pump.

Figure 7.14: The Trans-Alaska Pipeline System, which inspired this problem. It was built in the 1970s, at tremendous financial, political and social cost.

Before you start building the pipeline, the operator would like to know how the system would perform at half-capacity (i.e. with half the volume flow).

7.6.6. If none of the other input data changes, what is the new pumping power?

7.6.7. Propose one reason why in practice, the pumping power may be higher than you just calculated (briefly justify your answer, e.g. in 30 words or less).

7.7 Pump with pipe expansion

A pump is used to carry a volume flow of 200 L s^{-1} from one large water reservoir to another (fig. 7.15). The altitude of the water surface in both reservoirs is the same.

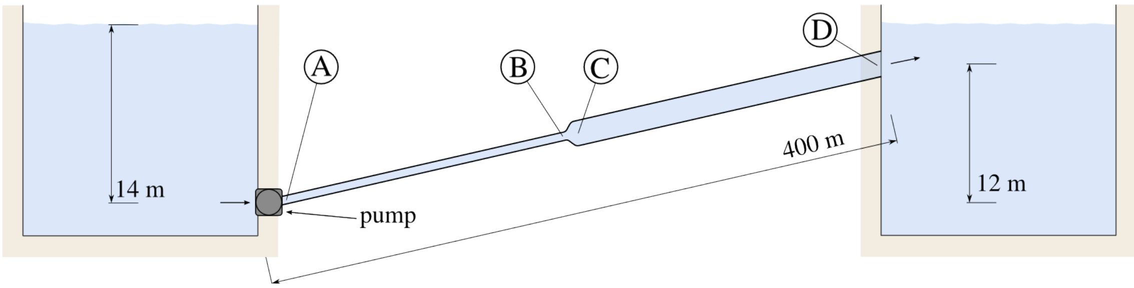


Figure 7.15: Layout of the water pipe. For clarity, the diameter of the pipe and the vertical scale are exaggerated.

Figure CC-0 *Olivier Cleynen*

The pipe connecting the reservoirs is made of concrete (ϵ = 0,25 mm); it has a diameter of 50 cm on the first half, and 100 cm on the second half. In the middle, the conical expansion element induces a loss coefficient of 0,8. At the outlet (at point D), the pressure is approximately equal to the corresponding hydrostatic pressure in the outlet tank.

The inlet is 14 m below the surface. The total pipe length is 400 m; the altitude change between inlet and outlet is 12 m.

7.7.1. Represent qualitatively (that is to say, showing the main trends, but without displaying accurate values) the water pressure as a function of pipe distance, when the pump is turned off.

7.7.2. On the same graph, represent qualitatively the water pressure when the pump is switched on.

7.7.3. What is the water pressure at points A, B, C and D?

7.8 Piping and power of a water turbine

from 2018-07 final examination

A water turbine is installed between two reservoirs in order to extract power from the flow of water. The water is guided to the turbine through a pipe, as shown in figure 7.16.

The pipe is made of reinforced concrete, with a total length L = 0,8 km, a diameter D = d = 1,2 m, and an interior surface roughness ϵ = 6 mm. The pipe has altitude variations along its length, as indicated in figure 7.16. The turbine is designed to handle 5 000 L s^{-1} of water at 20 °C.

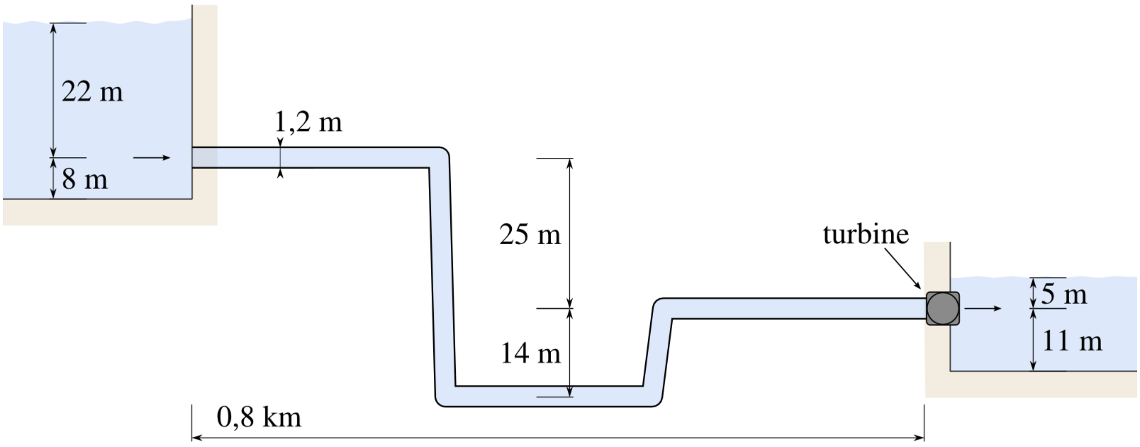


Figure 7.16: Layout of the water pipe. For clarity, the vertical scale and the diameter of the pipe are greatly exaggerated.

7.8.1. On a diagram, represent qualitatively (i.e. without numerical data) the pressure distribution along the length of the pipe.

7.8.2. What is the pressure drop due to friction losses generated by the water flow in the pipe?

7.8.3. What is the power developed by the turbine?

7.8.4. What would be the new power developed by the turbine if the volume flow was divided by two?

7.9 Politically incorrect fluid mechanics

non-examinable

In spite of the advice of their instructor, a group of students attempts to apply fluid mechanics to incommendable activities. Their objective is to construct a drinking straw piping system that can mix a drink of vodka and tonic water in the correct proportions (fig. 7.17). They use a "*Strawz*" kit of connected drinking straws, two bottles, and a glass full of ice and liquid water to cool the mix.

For simplicity, the following information is assumed about the setup:

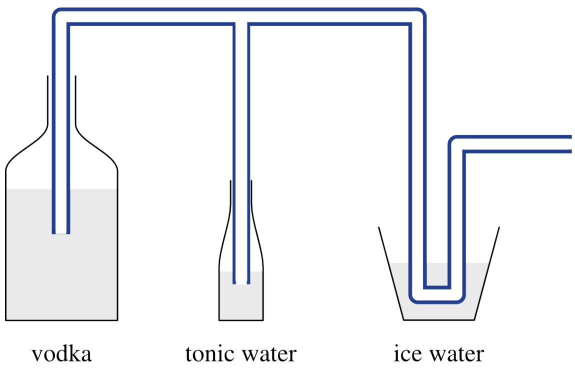


Figure 7.17: Conceptual sketch of a student experiment.

Percentage of alcohol by weight	Viscosity in centipoise
0	1,005
10	1,538
20	2,183
30	2,71
40	2,91
50	2,87
60	2,67
70	2,37
80	2,008
90	1,61
100	1,2

Table 7.1: Viscosity of a mix of ethanol and water at 20 °C.

data from Bingham, Trans. Chem. Soc, 1902, 81, 179.

- Vodka is modeled as 40 % pure alcohol (ethanol) with 60 % water by volume;
- Ethanol density is 0,8 $kg\,m^{-3}$;
- Tonic water is modeled as pure water;
- The viscosity of water and alcohol mixes is described in table 7.1 (use the nearest relevant value);
- The pipe bends induce a loss coefficient factor $K_{L\text{bend}} = 0{,}5$ each;
- The pipe T-junction induces a loss coefficient factor $K_L = 0{,}3$ in the line direction and 1 in the branching flow;
- The pipe has inner diameter $D = 3$ mm and roughness $\eta = 0{,}0025$ mm.

The students wish to obtain the correct mix: one quarter vodka, three quarters tonic water. For given levels of liquid in the bottles, is there a straw pipe network configuration that will yield the correct mix, and if so, what is it?

Answers

7.3 1) Calculating inlet density with the perfect gas model, $[\mathrm{Re}]_D = 14\,263$ (turbulent), a Moody diagram read gives $f \approx 0{,}029$, so $\Delta p_{\text{friction}} = -1\,292{,}6\,\mathrm{Pa} = -0{,}0129\,\mathrm{bar}$.

7.4 $|\Delta p_{\text{alt.}}| = \rho g(26 - 8 + 5 - 7) = 1{,}57\,\mathrm{bar}$ and $|\Delta p_{\text{friction}}| = 51{,}87\,\mathrm{bar}$: $\dot{W}_{\text{pump}} = 5{,}345\,\mathrm{MW}$.

7.5 1) $D_{\min} = 0{,}46\,\mathrm{m}$
2) Follow the process used p. 139 to go from eq. 7/10 to eq. 7/13: the only difference is the use of cylindrical (instead of rectangular) coordinates.
3) $|\Delta p|_{\text{loss}} = 0{,}75\,\mathrm{Pa}$
4) $\dot{W} = 0{,}63\,\mathrm{mW}$
5) $|\Delta p|_{\text{loss 2}} = 4{,}82\,\mathrm{kPa}$ (8 000 times more)
6) $\dot{W}_2 = 4{,}2\,\mathrm{W}$

7.6 1) Approximately 12 days 18 hours
2) $\dot{W}_{\text{total}} = 10{,}02\,\mathrm{MW}$
3) $\Delta L_{\text{flat terrain}} = 213\,\mathrm{km}$
4) $\Delta L_{\text{ascending terrain}} = 11{,}96\,\mathrm{km}$
7) Hint: in normal operation, what happens with the 10 MW of power — in which form is this energy converted?

7.7 3) $\Delta p_{f\,\mathrm{A}\to\mathrm{B}} = -3\,735\,\mathrm{Pa}$, $\Delta p_{\mathrm{B}\to\mathrm{C}} = +71\,\mathrm{Pa}$ (the sum of losses due to friction and gains due to decrease in kinetic energy), $\Delta p_{f\,\mathrm{C}\to\mathrm{D}} = -117\,\mathrm{Pa}$. Add hydrostatic pressure changes, working backwards from D to A, to obtain pressure in all four points.

7.8 2) $\Delta p_{\text{friction losses}} = -2{,}0196\,\mathrm{bar}$
3) $\dot{W}_{\text{turbine}} = -1{,}0503\,\mathrm{MW}$
4) The power will decrease by 14 %

7.9 The author cannot remember which exercise you are referring to.