Fluid Mechanics
Chapter 6 – Scale effects

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These lecture notes are based on textbooks by White [12], Çengel & al.[15], and Munson & al.[17].

6.1 Motivation

In this chapter, we use the tools we derived in chapters 3 and 4 to analyze scale effects in fluid mechanics. This study should allow us to answer two questions:

- How can a flow be adequately reproduced in a smaller or larger scale?
- How do forces and powers change when the flow is scaled?

6.2 Scaling forces

6.2.1 Force coefficients

Let us begin as a designer working for an automobile manufacturer. We wish to investigate the air flow around a full-size car, using a small model in a wind tunnel. The question we seek to answer is: if we succeed in "reproducing the exact real flow" in the wind tunnel, how large (or small) will the aerodynamic forces be on the model?

For this, we wish to have a sense of how flow-induced forces scale when fluid flows are scaled. A look back on chapter 3, and in particular eq. 3/8 p.65, reminds us that for a steady flow through a given control volume, the net force induced on the fluid is expressed by:

$$\vec{F}_{\text{net}} = \iint_{\text{CS}} \rho \vec{V} \ (\vec{V}_{\text{rel}} \cdot \vec{n}) \, \mathrm{d}A = \sum_{\text{net}} \left\{ (\rho |V_\perp| A) \vec{V} \right\} \qquad (6/1)$$

This equation tells us that the norm of the force vector $\vec{F}_{\text{net}}$ is directly related to the norm of the vector $\Sigma\, \rho |V_\perp| A \vec{V}$. In the selection of a scale by which to measure F_{net}, it is therefore sensible to include a term proportional to the the density ρ, a term proportional to the area A, and a term proportional to the square of velocity V. This "scale of fluid-induced force" is conventionally measured using the *force coefficient* C_F:

$$C_F \equiv \frac{F}{\frac{1}{2}\rho S V^2} \tag{6/2}$$

where F is the considered fluid-induced force (N);
ρ is a reference fluid density (kg m^{-3});
S is a reference surface area (m^2);
and V is a reference velocity (m s^{-1}).

In effect, the force coefficient relates the magnitude of the force exerted by the fluid on an object (F) to the rate of flow of momentum towards the object ($\rho S V^2 = \dot{m} V$).

It is worth making a few remarks about this equation. First, it is important to realize that eq. 6/2 is a *definition*: while the choice of terms is guided by physical principles, it is not a physical law in itself, and there are no reasons to expect C_F (which has no dimension, and thus no unit) to reach any particular value in any given case. The choice of terms is also worth commenting:

- F can be any fluid-induced force; generally we are interested in quantifying either the *drag* F_{D} (the force component parallel to the free-stream velocity) or the *lift* F_{L} (force component perpendicular to the free-stream velocity);
- the reference area S is chosen arbitrarily. It matters only that this area grow and shrink in proportion to the studied flow case. In the automobile industry, it is customary to choose the vehicle frontal area as a reference, while in the aeronautical industry the top-view wing area is customarily used. The square of a convenient reference length L can also be chosen instead;
- the choice of reference velocity V and density ρ is also arbitrary, since both these properties may vary in time and space within the studied fluid flow case. It matters merely that the chosen reference values are representative of the case; typically the free-stream (faraway) conditions V_∞ and ρ_∞ are used;
- the term 1/2 in the denominator is a purely arbitrary and conventional value.

Force coefficients are meaningful criteria to compare and relate what is going on in the wind tunnel and on the full-size object: in each case, we scale the measured force according to the relevant local flow conditions.

6.2.2 Power coefficient, and other coefficients

During our investigation of the flow using a small-scale model, we may be interested in measuring not only forces, but also other quantities such as power — this would be an important parameter, for example, when studying a pump, an aircraft engine, or a wind turbine.

Going back once again to chapter 3, and in particular eq. 3/15 p.68, we recall that power gained or lost by a fluid flowing steadily through a control volume could be expressed as:

$$\frac{\mathrm{d}E_{\text{sys}}}{\mathrm{d}t} = \iint_{\text{CS}} \rho\ e\ (\vec{V}_{\text{rel}} \cdot \vec{n})\ \mathrm{d}A = \sum_{\text{net}} \{(\rho|V_\perp|A)e\} \tag{6/3}$$

in which the specific energy e grows proportionally to $\frac{1}{2}V^2$.

Thus, the power gained or lost by the fluid is directly related to the magnitude of the scalar $\Sigma\ \rho|V_\perp|AV^2$. A meaningful "scale" for the power of a machine can therefore be the amount of energy in the fluid that is made available to it every second, a quantity that grows proportionally to ρAV^3.

This "scale of fluid flow-related power" is conventionally measured using the *power coefficient* C_{P}:

$$C_{\text{P}} \equiv \frac{\dot{W}}{\frac{1}{2}\rho S V^3} \tag{6/4}$$

where $\dot{W}$ is the power added to or subtracted from the fluid (W).

Other quantities related to fluid flow, such as pressure loss or shear friction, are measured using similarly-defined coefficients. We have already used the *pressure loss coefficient* $K_L \equiv |\Delta p|/\frac{1}{2}\rho V_{\text{av.}}^2$ in chapter 5 (eq. 5/23 p.114), and we shall soon use the *shear coefficient* c_f in the coming chapter (eq. 7/5 p.151).

6.2.3 Physical similarity

In the paragraphs above, we have already hinted that under certain conditions, the force and power coefficients in experiments of differing scales may have the same value. This happens when we have attained *physical similarity*. Several levels of similarity may be achieved between two experiments:

- geometrical similarity (usually an obvious requirement), by which we ensure similarity of shape between the solid boundaries, which are then all related by a scale factor r;
- kinematic similarity, by which we ensure that the motion of all solid and fluid particles is reproduced to scale — thus that for a given time scale ratio r_t the relative positions are scaled by r, the relative velocities by r/r_t and the relative accelerations by $(r/r_t)^2$;
- dynamic similarity, by which we ensure that not only the movement, but also the *forces* are all scaled by the same factor.

In more complex cases, we may want to observe other types of similarity, such as thermal or chemical similarity (for which ratios of temperature or concentrations become important).

The general rule that we follow is that *the measurements in two experiments can be meaningfully compared if the experiments are physically similar.* A direct consequence for our study of fluid mechanics is that *in two different experiments, the force and power coefficients will be the same if the flows are physically similar.*

In the design of experiments in fluid mechanics, considerable effort is spent to attain at least partial physical similarity. Once this is done, measurements can easily be extrapolated from one case to the other because all forces, velocities, powers, pressure differences etc. can be connected using coefficients.

For example, a flow case around a car A may be studied using a model B. *If dynamic similarity is maintained* (and that is by no means an easy task!), then the flow dynamics will be identical. The drag force $F_{\text{D B}}$ measured on the model can then be compared to the force $F_{\text{D A}}$ on the real car, using coefficients: since $C_{F_{\text{D B}}} = C_{F_{\text{D A}}}$ we have:

$$F_{\text{D A}} = C_{F_{\text{D A}}} \frac{1}{2} \rho_A S_{\text{A}} V_{\text{A}}^2 = C_{F_{\text{D B}}} \frac{1}{2} \rho_A S_{\text{A}} V_{\text{A}}^2 = \frac{F_{\text{D B}}}{\frac{1}{2} \rho_B S_{\text{B}} V_{\text{B}}^2} \frac{1}{2} \rho_A S_{\text{A}} V_{\text{A}}^2$$
$$= \left(\frac{\rho_A}{\rho_B} \frac{S_{\text{A}}}{S_{\text{B}}} \frac{V_{\text{A}}^2}{V_{\text{B}}^2} \right) F_{\text{D B}}$$

It should be noted that even when perfect dynamic similarity is achieved, fluid-induced forces do not scale with other types of forces. For example, if the length of model B is 1/10th that of car A, then $S_{\text{B}} = 0{,}1^2 S_{\text{A}}$ and at identical fluid density and velocity, we would have $F_{\text{D B}} = 0{,}1^2 F_{\text{D A}}$. Nevertheless, the *weight* $F_{\text{W B}}$ of model B would not be one hundredth of the weight of car A. If the same materials were used to produce the model, weight would stay proportional to *volume*, not surface, and we would have $F_{\text{W B}} = 0{,}1^3 F_{\text{W A}}$. This has important consequences when weight has to be balanced by flow-induced forces such as aerodynamic lift on an airplane. It is also the reason why large birds such as condors or swans do not look like, and cannot fly as slowly as mosquitoes and bugs![1]

6.3 Scaling flows

We have just seen that when two flows are dynamically similar, their comparison is made very easy, as measurements made in one case can be transposed to the other with the use of coefficients. This naturally raises the question: how can we *ensure* that two flows are flows are dynamically similar? The answer comes from a careful re-writing of the Navier-Stokes equation: two flows are dynamically similar when all the relevant *flow coefficients* have the same value.

6.3.1 The non-dimensional Navier-Stokes equation

In this section we wish to obtain an expression for the Navier-Stokes equation for incompressible flow that allows for an easy comparison of its constituents. We start with the original equation, which we derived in chapter 4 as eq. 4/37 p.94:

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho (\vec{V} \cdot \vec{\nabla}) \vec{V} = \rho \vec{g} - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{V} \qquad (6/5)$$

What we would now like to do is *separate the geometry from the scalars* in this equation. This is akin to re-expressing each vector $\vec{A}$ as the multiple of its length A and a *non-dimensional* vector $\vec{A}^*$, which has the same direction as $\vec{A}$ but only unit length.

[1]These ideas are beautifully and smartly explored in Tennekes [6, 13].

In order to achieve this, we introduce a series of non-dimensional physical terms, starting with non-dimensional time t^*, defined as time t multiplied by the frequency f at which the flow repeats itself:

$$t^* \equiv f t \tag{6/6}$$

A flow with a very high frequency is highly unsteady, and the changes in time of the velocity field will be relatively important. On the other hand, flows with very low frequencies are quasi-steady. In all cases, as we observe the flow, non-dimensional time t^* progresses from 0 to 1, after which the solution is repeated.

We then introduce non-dimensional speed $\vec{V}^*$, a unit vector field equal to the speed vector field divided by its own (scalar field) length V:

$$\vec{V}^* \equiv \frac{\vec{V}}{V} \tag{6/7}$$

Pressure p is non-dimensionalized differently, since in fluid mechanics, it is the pressure *changes* across a field, not their absolute value, that influence the velocity field. For example, in eq. 6/5 $\vec{\nabla} p$ can be replaced by $\vec{\nabla}(p - p_\infty)$ (in which p_∞ can be any constant faraway pressure). Now, the pressure field $p - p_\infty$ is non-dimensionalized by dividing it by a reference pressure difference $p_0 - p_\infty$, obtaining:

$$p^* \equiv \frac{p - p_\infty}{p_0 - p_\infty} \tag{6/8}$$

If p_0 is taken to be the maximum pressure in the studied field, then p^* is a scalar field whose values can only vary between 0 and 1.

Non-dimensional gravity $\vec{g}^*$ is simply a unit vector equal to the gravity vector $\vec{g}$ divided by its own length g:

$$\vec{g}^* \equiv \frac{\vec{g}}{g} \tag{6/9}$$

And finally, we define a new operator, the *non-dimensional del* $\vec{\nabla}^*$,

$$\vec{\nabla}^* \equiv L \, \vec{\nabla} \tag{6/10}$$

which ensures that vector fields obtained with a non-dimensional gradient, and the scalar fields obtained with a non-dimensional divergent, are "scaled" by a reference length L.

These new terms allow us to replace the constituents of equation 6/5 each by a non-dimensional "unit" term multiplied by a scalar term representing its length or value:

$$t = \frac{t^*}{f}$$
$$\vec{V} = V \, \vec{V}^*$$
$$p - p_\infty = p^* \, (p_0 - p_\infty)$$
$$\vec{g} = g \, \vec{g}^*$$
$$\vec{\nabla} = \frac{1}{L} \vec{\nabla}^*$$

Now, inserting these in equation 6/5, and re-arranging, we obtain:

$$\rho \frac{\partial}{\partial \frac{1}{f} t^*} \left(V\vec{V}^* \right) + \rho \left(V\vec{V}^* \cdot \frac{1}{L}\vec{\nabla}^* \right) \left(V\vec{V}^* \right) = \rho g \, \vec{g}^* - \frac{1}{L}\vec{\nabla}^* \left[p^* \left(p_0 - p_\infty \right) + p_\infty \right] + \mu \frac{\vec{\nabla}^{*2}}{L^2} \left(V\vec{V}^* \right)$$

$$\rho V f \frac{\partial \vec{V}^*}{\partial t^*} + \rho V V \frac{1}{L} \left(\vec{V}^* \cdot \vec{\nabla}^* \right) \vec{V}^* = \rho g \, \vec{g}^* - \frac{1}{L}\vec{\nabla}^* \left[p^* \left(p_0 - p_\infty \right) \right] + \mu V \frac{1}{L^2} \vec{\nabla}^{*2} \vec{V}^*$$

$$[\rho V f] \; \frac{\partial \vec{V}^*}{\partial t^*} + \left[\frac{\rho V^2}{L} \right] \; \left(\vec{V}^* \cdot \vec{\nabla}^* \right) \vec{V}^* = [\rho g] \; \vec{g}^* - \left[\frac{p_0 - p_\infty}{L} \right] \; \vec{\nabla}^* p^* + \left[\frac{\mu V}{L^2} \right] \; \vec{\nabla}^{*2} \vec{V}^* \tag{6/11}$$

In equation 6/11, the terms in brackets each appear in front of non-dimensional (unit) vectors. These bracketed terms all have the same dimension, i.e. $\text{kg}\,\text{m}^{-2}\,\text{s}^{-2}$. Multiplying each by $\frac{L}{\rho V^2}$ (of dimension $\text{m}^2\,\text{s}^2\,\text{kg}^{-1}$), we obtain a purely non-dimensional equation:

$$\left[\frac{fL}{V} \right] \; \frac{\partial \vec{V}^*}{\partial t^*} + [1] \; \left(\vec{V}^* \cdot \vec{\nabla}^* \right) \vec{V}^* = \left[\frac{gL}{V^2} \right] \; \vec{g}^* - \left[\frac{p_0 - p_\infty}{\rho V^2} \right] \; \vec{\nabla}^* p^* + \left[\frac{\mu}{\rho V L} \right] \; \vec{\nabla}^{*2} \vec{V}^* \tag{6/12}$$

Equation 6/12 does not bring any information on top of the original incompressible Navier-Stokes equation (eq. 6/5). Instead, it merely separates it into two distinct parts. The first is a scalar field (purely numbers, and noted in brackets), which indicates the magnitude of the acceleration field. The other is a unit vector field (a field of oscillating vectors, all of length one, and noted with stars), which represents the geometry (direction) of the acceleration field. In this form, we can more easily observe and quantify the weight of the different terms relative to one another. This is the role of the flow parameters.

6.3.2 The flow parameters of Navier-Stokes

From here on, we convene to call the terms written in brackets in eq. 6/12 *flow parameters*, and label them with the following definitions:

$$[\text{St}] \equiv \frac{f\,L}{V} \tag{6/13}$$

$$[\text{Eu}] \equiv \frac{p_0 - p_\infty}{\rho\,V^2} \tag{6/14}$$

$$[\text{Fr}] \equiv \frac{V}{\sqrt{g\,L}} \tag{6/15}$$

$$[\text{Re}] \equiv \frac{\rho\,V\,L}{\mu} \tag{6/16}$$

This allows us to re-write eq. 6/12 as:

$$[\text{St}] \; \frac{\partial \vec{V}^*}{\partial t^*} + [1] \; \vec{V}^* \cdot \vec{\nabla}^* \vec{V}^* = \left[\frac{1}{\text{Fr}^2} \right] \; \vec{g}^* - [\text{Eu}] \; \vec{\nabla}^* p^* + \left[\frac{1}{\text{Re}} \right] \; \vec{\nabla}^{*2} \vec{V}^*$$

And finally, we arrive at the simple, elegant and formidable *non-dimensional incompressible Navier-Stokes equation* expressed with flow parameters:

$$[\mathrm{St}]\ \frac{\partial \vec{V}^*}{\partial t^*} + [1]\ \vec{V}^* \cdot \vec{\nabla}^* \vec{V}^* = \frac{1}{[\mathrm{Fr}]^2}\ \vec{g}^* - [\mathrm{Eu}]\ \vec{\nabla}^* p^* + \frac{1}{[\mathrm{Re}]}\ \vec{\nabla}^{*2} \vec{V}^* \tag{6/17}$$

Equation 6/17 is an incredibly potent tool in the study of fluid mechanics, for two reasons:

1. It allows us to quantify the **relative weight** of the different terms in a given flow.
   In this way, it serves as a compass, helping us determine which terms can safely be neglected in our attempts to find a particular flow solution, merely by quantifying the four parameters [St], [Eu], [Fr], and [Re].

2. It allows us to obtain **dynamic similarity** between two flows at different scales (fig. 6.1).
   In order that a fluid flow be representative of another flow (for example in order to simulate airflow around an aircraft with a model in a wind tunnel), it must generate the same vector field $\vec{V}^*$. In order to do this, we must generate an incoming flow with the same four parameters [St], [Eu], [Fr], and [Re].

Let us therefore take the time to explore the signification of these four parameters.

The Strouhal number $[\mathrm{St}] \equiv \frac{f\ L}{V}$ (eq. 6/13) quantifies the influence of unsteadiness effects over the acceleration field. It does this by evaluating the importance of the representative frequency f at which the flow pattern repeats itself. Very high frequencies denote highly unsteady flows. When the frequency is very low, [St] is very small; the flow is then quasi-steady and it can be solved at a given moment in time as if it was entirely steady.

The Euler number $[\mathrm{Eu}] \equiv \frac{p_0 - p_\infty}{\rho\ V^2}$ (eq. 6/14) quantifies the influence of the pressure gradient over the acceleration field. It does this by comparing the largest relative pressure $p_0 - p_\infty$ to the flow of momentum in the flow field. The greater [Eu] is, and the more the changes in the velocity field $\vec{V}$ are likely to be caused by pressure gradients rather than viscosity, convection or unsteadiness.

The Froude number $[\mathrm{Fr}] \equiv \frac{V}{\sqrt{g\ L}}$ (eq. 6/15) quantifies the relative importance of gravity effects. In practice, gravity effects only play an important role in free surface flows, such as waves on the surface of a water reservoir. In most other cases, gravity contributes only to a hydrostatic effect, which has little influence over the velocity field.

The Reynolds number $[\mathrm{Re}] \equiv \frac{\rho\ V\ L}{\mu}$ (eq. 6/16, also eqs. 0/27 p.21 & 5/25 p.114) quantifies the influence of viscosity over the acceleration field. It does this by comparing the magnitude of inertial effects ($\rho V L$) to viscous effects (μ). When [Re] is very large, viscosity plays a negligible role and the velocity field is mostly dictated by the inertia of the fluid. We return to the significance of the Reynolds number in §6.5 below.

The Mach number [Ma] ≡ V/a (eq. 0/10 p.15) compares the flow velocity V with that of the molecules within the fluid particles (the speed of sound a). [Ma] does not appear in equation 6/17 because from the start in chapter 4, we chose to restrict ourselves to non-compressible flows. If we hadn't, an additional term would exist on the right-hand side, $+\frac{1}{3}\mu\vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}\right)$, which is null when the divergent of velocity $\vec{\nabla}\cdot\vec{V}$ is zero (which is always so at low to moderate flow speeds).
We shall explore the significance of the Mach number in chapter 9.

These five flow parameters should be thought of *scalar fields* within the studied flow domain: there is one distinct Reynolds number, one Mach number etc. for each point in space and time. Nevertheless, when describing fluid flows, the custom is to choose for each parameter *a single representative value* for the whole flow. For example, when describing pipe flow, it is customary to quantify a representative Reynolds number $[\text{Re}]_D$ based on the average flow velocity $V_{\text{av.}}$ and the pipe diameter D (as we have seen with eq. 5/26 p. 116), while the representative Reynolds number for flow over an aircraft wing is often based on the free-stream velocity V_∞ and the wing chord length c. Similarly, the flight Mach number $[\text{Ma}]_\infty$ displayed on an aircraft cockpit instrument is computed using the relative free-stream air speed V_∞ and the free-stream speed of sound a_∞, rather than particular values measured closer to the aircraft.

6.3.3 Scaling flows

Although the above mathematical treatment may be intimidating, it leads us to a rather simple and useful conclusion, which can be summarized as follows.
When studying one particular fluid flow around or within an object, it may be useful to construct a model that allows for less expensive or more comfortable measurements (fig. 6.1). In that case, we need to make sure that the fluid flow on the model is representative of the real case. This can be achieved only if the relevant flow parameters have the same value in both cases. If this can be achieved —although in practice this can rarely be achieved exactly!— then the force and power coefficients also keep the same value. The scale effects are then fully-controlled and the model measurements are easily extrapolated to the real case.

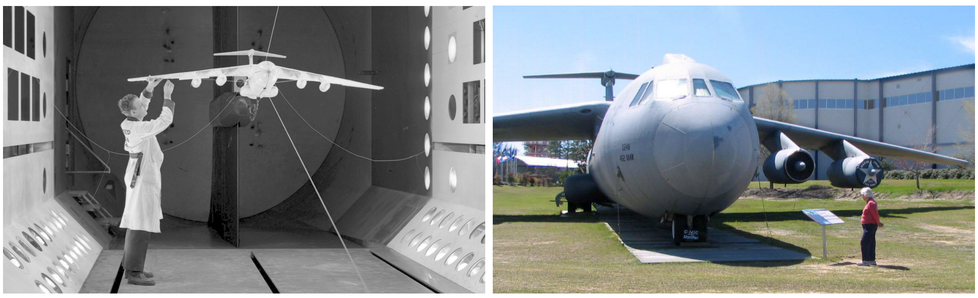

Figure 6.1 – In order for the flow around a wind tunnel model to be representative of the flow around the real-size aircraft (here, a 48 m-wide Lockheed C-141 Starlifter), dynamic similarity must be obtained. The value of all flow parameters must be kept identical. This is not always feasible in practice.

Wind tunnel photo by NASA (public domain)
Full-size aircraft photo CC-BY *by Peter Long*

6.4 Flow parameters as force ratios

This topic is well covered in Massey [4]

Instead of the above mathematical approach, the concept of flow parameter can be approached by *comparing forces* in fluid flows.

Fundamentally, understanding the movement of fluids requires applying Newton's second law of motion: the sum of forces which act upon a fluid particle is equal to its mass times its acceleration. We have done this in an aggregated manner with integral analysis (in chapter 3, eq. 3/8 p.65), and then in a precise and all-encompassing way with differential analysis (in chapter 4, eq. 4/37 p.94). With the latter method, we obtain complex mathematics suitable for numerical implementation, but it remains difficult to obtain rapidly a quantitative measure for what is happening in any given flow.

In order to obtain this, an engineer or scientist can use force ratios. This involves comparing the magnitude of a type of force (pressure, viscous, gravity) either with another type of force, or with the mass-times-acceleration which a fluid particle is subjected to as it travels. We are not interested in the absolute value of the resulting ratios, but rather, in having a measure of the parameters that influence them, and being able to compare them across experiments.

6.4.1 Acceleration vs. viscous forces: the Reynolds number

The net sum of forces acting on a particle is equal to its mass times its acceleration. If a representative length for the particle is L, the particle mass grows proportionally to the product of its density ρ and its volume L^3. Meanwhile, its acceleration relates how much its velocity V will change over a time interval Δt: it may be expressed as a ratio $\Delta V/\Delta t$. In turn, the time interval Δt may be expressed as the representative length L divided by the velocity V, so that the acceleration may be represented as proportional to the ratio $V\Delta V/L$. Thus we obtain:

$$|\text{net force}| = |\text{mass} \times \text{acceleration}| \sim \rho L^3 \frac{V\Delta V}{L}$$
$$|\vec{F}_{\text{net}}| \sim \rho L^2 V \Delta V$$

We now observe the viscous force acting on a particle: it is proportional to the the shear effort and a representative acting surface L^2. The shear can be modeled as proportional to the viscosity μ and the rate of strain, which will grow proportionally to $\Delta V/L$. We thus obtain a crude measure for the magnitude of the shear force:

$$|\text{viscous force}| \sim \mu \frac{\Delta V}{L} L^2$$
$$|\vec{F}_{\text{viscous}}| \sim \mu \Delta V L$$

The magnitude of the viscous force can now be compared to the net force:

$$\frac{|\text{net force}|}{|\text{viscous force}|} \sim \frac{\rho L^2 V \Delta V}{\mu \Delta V L} = \frac{\rho V L}{\mu} = [\text{Re}] \qquad (6/18)$$

and we recognize the ratio as the Reynolds number (6/16). We thus see that the Reynolds number can be interpreted as the inverse of the influence of viscosity. The larger [Re] is, and the smaller the influence of the viscous forces will be on the trajectory of fluid particles.

6.4.2 Acceleration vs. gravity force: the Froude number

The weight of a fluid particle is equal to its mass, which grows with ρL^3, multiplied by gravity g:

$$|\text{weight force}| = |\vec{F}_\text{W}| \sim \rho L^3 g$$

The magnitude of this force can now be compared to the net force:

$$\frac{|\text{net force}|}{|\text{weight force}|} \sim \frac{\rho L^2 V^2}{\rho L^3 g} = \frac{V^2}{Lg} = [\text{Fr}]^2 \tag{6/19}$$

and here we recognize the square of the Froude number (6/15). We thus see that the Froude number can be interpreted as the inverse of the influence of weight on the flow. The larger [Fr] is, and the smaller the influence of gravity will be on the trajectory of fluid particles.

6.4.3 Acceleration vs. elastic forces: the Mach number

In some flows called *compressible flows* the fluid can perform work on itself, and and the fluid particles then store and retrieve energy in the form of changes in their own volume. In such cases, fluid particles are subject to an *elastic force* in addition to the other forces. We can model the pressure resulting from this force as proportional to the bulk modulus of elasticity K of the fluid (formally defined as $K \equiv \rho\ \partial p/\partial\rho$); the elastic force can therefore be modeled as proportional to KL^2:

$$|\text{elasticity force}| = |\vec{F}_\text{elastic}| \sim KL^2$$

The magnitude of this force can now be compared to the net force:

$$\frac{|\text{net force}|}{|\text{elasticity force}|} \sim \frac{\rho L^2 V^2}{KL^2} = \frac{\rho V^2}{K}$$

This ratio is known as the Cauchy number; it is not immediately useful because the value of K in a given fluid varies considerably not only according to temperature, but also according to the type of compression undergone by the fluid: for example, it grows strongly during brutal compressions.

During isentropic compressions and expansions (isentropic meaning that the process is fully reversible, i.e. without losses to friction, and adiabatic, i.e. without heat transfer), we will show in chapter 9 (with eq. 9/7 p.195) that the bulk modulus of elasticity is proportional to the square of the speed of sound a:

$$K|_\text{reversible} = a^2 \rho \tag{6/20}$$

The Cauchy number calibrated for isentropic evolutions is then

$$\frac{|\text{net force}|}{|\text{elasticity force}|_{\text{reversible}}} \sim \frac{\rho V^2}{K} = \frac{V^2}{a^2} = [\text{Ma}]^2 \qquad (6/21)$$

and here we recognize the square of the Mach number (0/10). We thus see that the Mach number can be interpreted as the influence of elasticity on the flow. The larger [Ma] is, and the smaller the influence of elastic forces will be on the trajectory of fluid particles.

6.4.4 Other force ratios

The same method can be applied to reach the definitions for the Strouhal and Euler numbers given in §6.3. Other numbers can also be used which relate forces that we have ignored in our study of fluid mechanics. For example, the relative importance of surface tension forces or of electromagnetic forces are quantified using similarly-constructed flow parameters.
In some applications featuring rotative motion, such as flows in centrifugal pumps or planetary-scale atmospheric weather, it may be convenient to apply Newton's second law in a rotating reference frame. This results in the appearance of new reference-frame forces, such as the Coriolis or centrifugal forces; their influence can then be studied using additional flow parameters.

In none of those cases can flow parameters give enough information to predict solutions. They do, however, provide quantitative data to indicate which forces are relevant in which places: this not only helps us understand the mechanisms at work, but also distinguish the negligible from the influential, a key characteristic of efficient scientific and engineering work.

6.5 The Reynolds number in practice

Among the five non-dimensional parameters described above, the *Reynolds number* [Re] is by far the most relevant in the study of most fluid flows, and it deserves a few additional remarks. As we have seen, the Reynolds number is a measure of how little effect the viscosity has on the time-change of the velocity vector field:

- With low [Re], the viscosity μ plays an overwhelmingly large role, and the velocity of fluid particles is largely determined by that of their own neighbors;
- With high [Re], the momentum ρV of the fluid particles plays a more important role than the viscosity μ, and the inertia of fluid particles affects their trajectory much more than the velocity of their neighbors.

In turn, this gives the Reynolds number a new role in characterization of flows: it can be thought of as **the likeliness of the flow being turbulent over the length** L. Indeed, from a kinematic point of view, viscous effects are highly stabilizing: they tend to harmonize the velocity field and smooth out disturbances. On the contrary, when these effects are overwhelmed by inertial effects, velocity non-uniformities have much less opportunity to dissipate, and at the slightest disturbance the flow will become and remain turbulent (fig. 6.2). This is the reason why the quantification of a representative Reynolds number is often the first step taken in the study of a fluid flow.

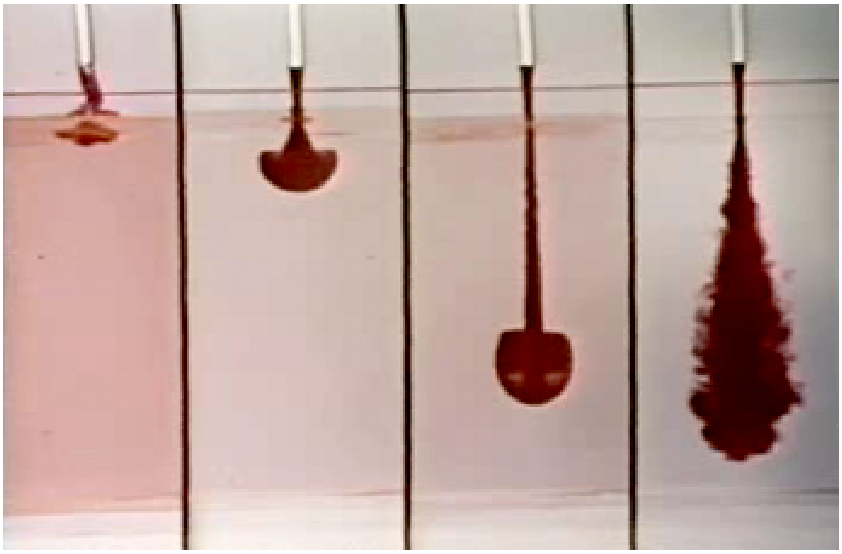

Figure 6.2 – A viscous opaque fluid is dropped into a clearer receiving static fluid with identical viscosity. The image shows four different experiments photographed after the same amount of time has elapsed. The viscosity is decreased from left to right, yielding Reynolds numbers of 0,05, 10, 200 and 3 000 respectively.
As described in eq. 6/18, low Reynolds number indicate that viscous effects dominate the change in time of the flow field. As the Reynolds number increases, the nature of the velocity field changes until it becomes clearly turbulent.

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A screen capture from film Low Reynolds Number Flow at http://web.mit.edu/hml/ncfmf.html