Fluid Mechanics
Chapter 4 – Effects of pressure
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4.1 Motivation

In fluid mechanics, only three types of forces apply to fluid particles: forces due to gravity, pressure, and shear. This chapter focuses on pressure (we will address shear in chapter 5), and should allow us to answer two questions:

• How is the effect of pressure described and quantified?
• What are the pressure forces generated on walls by static fluids?

4.2 Pressure forces on walls

4.2.1 Magnitude of the pressure force

What is the force with which a fluid pushes against a wall?

When the pressure \( p \) exerted is uniform and the wall is flat, the resulting force \( F \) is easily calculated:

\[
F_{\text{pressure}} = p_{\text{uniform}} \cdot S_{\text{flat wall}}
\]  

(4/1)

When the fluid pressure \( p \) is not uniform (for example, as depicted on the right side of the wall in figure 4.1), the situation is more complex: the force must be obtained by integration. The surface is split in infinitesimal portions of area \( dS \), and the corresponding forces are summed up as:

\[
F_{\text{pressure}} = \int_S dF_{\text{pressure}} = \int_S p \, dS
\]  

(4/2)

for a flat surface. where the \( S \)-integral denotes an integration over the entire surface.
4.2 Pressure and Moment of Pressure Forces

What is required to calculate the scalar \( F \) in eq. 4/2 is an expression of \( p \) as a function of \( S \). In a static fluid, this expression will often be easy to find, as we see later on. Typically, in two dimensions \( x \) and \( y \) we re-write \( dS \) as \( dS = dx\,dy \) and we may then proceed with the calculation starting from

\[
F_{\text{pressure}} = \iint p(x,y) \, dx \, dy \quad (4/3)
\]

The above equations work only for a flat surface. When we consider a two- or three-dimensional object immersed in a fluid with non-uniform pressure, the integration must be carried out with vectors. We will not attempt this in this course, but the expression is worth writing out in order to understand how computational fluid dynamics (CFD) software will proceed with the calculation:

\[
\vec{F}_{\text{pressure}} = \int_S d\vec{F} = \int_S p \, \vec{n} \, dS \quad (4/4)
\]

where the \( S \)-integral denotes an integration over the entire surface; and \( \vec{n} \) is a unit vector describing, on each infinitesimal surface element \( dS \), the direction normal to the surface.

## 4.2.2 Position of the pressure force

We are often interested not only in the magnitude of the pressure force, but also its position. This position can be evaluated by calculating the magnitude of moment generated by the pressure forces about any chosen point \( X \). This moment \( \vec{M}_X \), using notation shown in fig. 4.2, is expressed as:

\[
\vec{M}_X = \int_S d\vec{M}_X = \int_S \vec{r}_{XF} \wedge d\vec{F} = \int_S \vec{r}_{XF} \wedge p \, \vec{n} \, dS \quad (4/5)
\]

where \( \vec{r}_{XF} \) is a vector expressing the position of each infinitesimal surface relative to point \( X \).

Much like eq. 4/4 above, this eq. 4/5 is easily implemented in a software algorithm but not very approachable on paper. In this course however, we want to study the simple case where the surface is flat, and where the
Figure 4.2 – Moment generated about an arbitrary point X by the pressure exerted on an arbitrary surface (left: perspective view; right: side view). The vector \( \vec{n} \) is a convention unit vector everywhere perpendicular to the infinitesimal surface \( dS \) considered.

reference point X is in the same plane as the surface. Equation 4/5 is then a great deal simpler, and we can calculate the magnitude \( M_X \) as:

\[
M_X = \int_S dM_X = \int_S r_{XF} \, dF = \int_S r_{XF} \, p \, dS \quad (4/6)
\]

for a flat surface, with X in the plane of the surface.

Once both \( F_{\text{pressure}} \) and \( M_{X\text{pressure}} \) have been quantified, the distance \( R_{XF} \) between point X and the application point of the net pressure force is easily computed:

\[
R_{XF} = \frac{M_{X\text{pressure}}}{F_{\text{pressure}}} \quad (4/7)
\]

### 4.3 Pressure fields in fluids

We approached the concept of pressure in chapter 1 (Basic flow quantities) with the notion that it represented force perpendicular to a given flat surface (eq. 1/15), for example a flat plate of area \( A \):

\[
p = \frac{F_A}{A} \quad (4/8)
\]

To appreciate the concept of pressure in fluid mechanics, we need to go beyond this equation.

#### 4.3.1 The direction of pressure

An important concept is that in continuum mechanics, the flat surface is imaginary. More precisely, a fluid is able to exert pressure not only on solid
surfaces, but also upon and within itself. In this context, we need to rework eq. 4/8 so that now pressure is defined as perpendicular force per area on an \textit{infinitesimally small} surface of fluid:

\[
p = \lim_{A \to 0} \frac{F_z}{A}
\]

Equation 4/9 may appear unsettling at first sight, because as \( A \) tends to zero, \( F_z \) also tends to zero; nevertheless, in any continuous medium, the ratio of these two terms tends to a single non-zero value.

This brings us to the second particularity of pressure in fluids: the pressure on either side of the infinitesimal flat surface is the same regardless of its orientation. In other words, \textit{pressure has no direction}: there is only one (scalar) value for pressure at any one point in space.

Thus, in a fluid, pressure applies not merely on the solid surfaces of its container, but also everywhere within itself. We need to think of pressure as a \textit{scalar property field} \( p(x,y,z,t) \).

### 4.3.2 Pressure on an infinitesimal volume

While pressure has no direction, it may not have the same value everywhere in a fluid, and so the \textit{gradient} (the space rate of change) of pressure may not be null. For example, in a static water pool, pressure is uniform in the two horizontal directions, but it increases along with depth.

Instead of a flat plate, let us now consider an infinitesimally small \textit{cube} within the fluid (fig. 4.3). Because the cube is placed in a scalar field, the pressure exerting on each of its six faces may be different. The net effect of pressure will therefore have \textit{three} components: one for each pair of opposing faces.

What are those three components? In the \( x \)-direction, the pressure on faces 1 and 4 act upon a surface of area \( d\,y\,d\,z \):

\[
F_{\text{net, pressure, } x} = d\,y\,d\,z \left[ p_1 - p_4 \right]
\]

Figure 4.3 – The pressure on each face of an infinitesimal volume may have a different value. The \textit{net} effect of pressure will depend on how the pressure varies in space. These changes are labeled \( dp_i \) in each of the \( i = x, y, z \) directions.

\( Figure \, CC-0 \, o.c. \)
We express \( p_4 - p_1 \) as the derivative of pressure in the \( x \)-direction \((\partial p/\partial x)\), multiplied by the distance \( dx \) which separates points 1 and 4, obtaining:

\[
F_{\text{net, pressure,}x} = dy \, dz \left[ -\frac{\partial p}{\partial x} \, dx \right] = dV \frac{-\partial p}{\partial x}
\]  

(4/10)

where \( dV = dx \, dy \, dz \) is the volume of the infinitesimal cube.

Now generalizing eq. 4/10 for the other two directions, we can write:

\[
F_{\text{net, pressure,}x} = dV \frac{-\partial p}{\partial x} \\
F_{\text{net, pressure,}y} = dV \frac{-\partial p}{\partial y} \\
F_{\text{net, pressure,}z} = dV \frac{-\partial p}{\partial z}
\]

This is tedious to write, but we recognize a pattern. And indeed, we introduce the concept of gradient, a mathematical operator, defined as so (see also Appendix A2 p.225):

\[
\vec{\nabla} = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}
\]  

(4/11)

With this cool new tool, we elegantly re-write the system above:

\[
\vec{F}_{\text{net, pressure}} = -dV \, \vec{\nabla} p
\]  

(4/12)

Finally, we obtain:

\[
\frac{1}{dV} \vec{F}_{\text{net, pressure}} = -\vec{\nabla} p
\]  

(4/13)

This last equation reads "the pressure force per unit volume is the opposite of the pressure gradient". It shows us that in any fluid and any situation, the force due to pressure points the opposite way of the pressure gradient. Thus, if a particle of any kind is “dropped” into a fluid flow, we can quantify in which direction, and with which magnitude, pressure (a scalar field) is going to “push” it. This is given by equation 4/13, which quantifies this effect as a vector field.

### 4.4 Special case: pressure in static fluids

#### 4.4.1 Forces in static fluids

Fluid statics is the study of fluids at rest, i.e. whose velocity field \( \vec{V} \) is everywhere null and constant:

\[
\begin{align*}
\vec{V} &= \vec{0} \\
\frac{\partial \vec{V}}{\partial t} &= \vec{0}
\end{align*}
\]  

(4/14)
We choose to study this type of problem now, because it makes for a conceptually and mathematically simple case with which we can start analyzing fluid flow and calculate fluid-induced forces.

What are the forces applying on an arbitrary particle in a static fluid?

- The force due to pressure is related to the pressure gradient: we just quantified this with eq. 4/13.
- The force due to shear is zero. We will indeed see in chapter 5 (Effects of shear) that shear efforts can be expressed as a function of viscosity and velocity. All ordinary fluids are unable to exert shear when static.
- The force due to gravity is easy to quantify: it is the mass \( m \) of the fluid multiplied by the gravity vector \( \vec{g} \).

In a moving fluid, the sum of these forces would add up to the mass of the particle times its acceleration. But in a static fluid, the velocity is zero and never changes. We can thus write:

\[
\vec{F}_{\text{net, pressure}} + \vec{F}_{\text{shear}} + \vec{F}_{\text{gravity}} = \vec{0}
\]

\[-d \nabla \vec{p} + \vec{0} + mg = \vec{0}, \quad -\nabla \vec{p} + \vec{0} + \rho \vec{g} = \vec{0} \]

We can finally rewrite this as:

\[
\begin{align*}
\nabla \vec{p} &= \rho \vec{g} \\
\text{(4/15)}
\end{align*}
\]

This is a very useful equation, which states that in a static fluid, the only parameter affecting pressure is gravity. More precisely, the fluid density times the gravity vector is equal to the change in space of the pressure.

We will see in chapter 6 (Prediction of fluid flows) that equation 4/15 is the specific case for a much larger general and powerful equation, the *Navier-Stokes equation*. But more on that later!

### 4.4.2 Pressure and depth

It is now easy to quantify pressure everywhere inside a static fluid.

Very often in studies of static fluids, the \( z \)-axis is oriented vertically, positive downwards. With this convention, there is no need for a vector equation to quantify pressure, and equation 4/15 becomes:

\[
\frac{dp}{dz} = \rho g
\]

(4/16)

The first consequence we draw from equation 4/16 is that in a static fluid (e.g. in a glass of water, in a swimming pool, in a calm atmosphere), pressure depends solely on height. Within a static fluid, at a certain altitude, we will measure the same pressure regardless of the surroundings (fig. 4.4).

How is pressure distributed within static liquid water bodies? The density of liquid water is approximately constant: \( \rho_{\text{water}} = 1000 \text{ kg m}^{-3} \). In a water
Figure 4.4 – Pressure at a given depth (or height) in a static fluid does not depend on the environment. Here, as long as the fluid remains static, $p_A = p_B = p_C = p_D$.

Figure CC-0 o.c.

reservoir, equation 4/16 becomes:

$$\left( \frac{dp}{dz} \right)_{\text{water}} = \rho_{\text{water}} g$$

$$\left( \frac{dp}{dz} \right)_{\text{water}} = 1000 \times 9.81 = 9.81 \cdot 10^3 \text{ Pa m}^{-1} = 9.81 \cdot 10^{-2} \text{ bar m}^{-1}$$

(4/17)

Therefore, in static water, pressure increases by approximately 0.1 bar/m as depth increases. For example, at a depth of 3 m, the pressure will be approximately 1.3 bar (which is the atmospheric pressure plus $\Delta z \times \frac{dp}{dz}$).

In the atmosphere, the situation is more complex, because the density $\rho_{\text{air}}$ of atmospheric air is not uniform. If we model atmospheric air as a perfect gas, once again orienting $z$ vertically downwards, we can express the pressure gradient as:

$$\left( \frac{dp}{dz} \right)_{\text{atm.}} = \rho_{\text{air}} g = \rho \frac{1}{T} \frac{g}{R}$$

(4/18)

This time, the space variation of pressure depends on pressure itself (and it is proportional to it). A quick numerical investigation for ambient temperature and pressure (1 bar, 15°C) yields:

$$\left( \frac{dp}{dz} \right)_{\text{atm. ambient}} = 1 \cdot 10^5 \times \frac{1}{288.15} \times \frac{9.81}{287}$$

$$= 11.86 \text{ Pa m}^{-1} = 1.186 \cdot 10^{-4} \text{ bar m}^{-1}$$

This rate (approximately 0.1 mbar/m) is almost a thousand times smaller than that of water (fig. 4.5).

Since the rate of pressure change depends on pressure, it also varies with altitude, and the calculation of pressure differences in the atmosphere is a little more complicated than for water.

If we focus on a moderate height change, it may be reasonable to consider that temperature $T$, the gravitational acceleration $g$ and the parameter $R$ (gas constant) are uniform. In this admittedly restrictive case, equation 4/18 can
be integrated as so:

\[
\frac{dp}{dz} = \frac{g}{RT_{\text{est}}} \cdot p
\]

\[
\int_{1}^{2} \frac{1}{p} \, dp = \frac{g}{RT_{\text{est}}} \cdot \int_{1}^{2} \, dz
\]

\[
\ln \frac{p_2}{p_1} = \frac{g}{RT_{\text{est}}} \cdot \Delta z
\]

\[
\frac{p_2}{p_1} = \exp \left[ \frac{g \Delta z}{RT_{\text{est}}} \right]
\]

(4/19)

Of course, air temperature varies significantly within the atmosphere (at moderate altitudes the change with altitude is approximately \(-6 \text{ K km}^{-1}\)). Adapting equation 4/19 for a uniform temperature gradient (instead of uniform temperature) is the subject of problem 4.7 p.84.

In practice, the atmosphere also sees large lateral pressure gradients (which are strongly related to the wind) and its internal fluid mechanics are complex and fascinating. Equation 4/19 is a useful and convenient model, but refinements must be made if precise results are to be obtained.

4.4.3 Buoyancy

Any solid body immersed within a fluid is subjected to pressure on its walls. When the pressure is not uniform (for example because the fluid is subjected to gravity, although this may not be the only cause), then the net force due to fluid pressure on the body walls will be non-zero.

When the fluid is purely static, this net pressure force is called \textit{buoyancy}. Since in this case, the only cause for the pressure gradient is gravity, the net pressure force is oriented upwards. The buoyancy force is completely independent from (and may or may not compensate) the object’s weight.

Since it comes from equation 4/15 that the variation of pressure within a fluid is caused solely by the fluid’s weight, we can see that the force exerted on an immersed body is equal to the weight of the fluid it replaces (that is to say, the weight of the fluid that would occupy its own volume were it not there). This relationship is sometimes named \textit{Archimedes’ principle}. The
force which results from the static pressure gradient applies to all immersed bodies: a submarine in an ocean, an object in a pressurized container, and of course, the reader of this document as presently immersed in the earth’s atmosphere.

![Figure 4.6](image.png)

Figure 4.6 – Immersion in a static fluid results in forces that depend on the body’s volume. They can be evidenced by the removal of the fluid (for example in a depressurized semi-spherical vessel).

*Figure CC-0 o.c.*