Fluid Mechanics
Chapter 4 – Predicting fluid flow

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These lecture notes are based on textbooks by White [12], Çengel & al.[15], and Munson & al.[17].

4.1 Motivation

In this chapter we assign ourselves the daunting task of describing the movement of fluids in an extensive manner. We wish to express formally, and calculate, the *dynamics of fluids* —the velocity field as a function of time— in any arbitrary situation. For this, we develop a methodology named *derivateive analysis*.

Video: pre-lecture briefing for this chapter

https://youtu.be/wOGlqykD3_E

Let us not shy away from the truth: the methods developed here are disproportionately complex in comparison to the solutions they allow us to derive by hand. Despite this, this chapter is extraordinarily important, for two reasons:

- derivative analysis allows us to formally *describe* and *relate* the key parameters that regulate fluid flow, and so is the key to developing an understanding of any fluid phenomenon, even when solutions cannot be derived;
- it is the backbone for *computational fluid dynamics* (CFD) in which approximate flow solutions are obtained using numerical procedures.

4.2 Eulerian description of fluid flow

4.2.1 Problem description

From now on, we wish to describe the velocity and pressure fields of a fluid with the highest possible resolution. For this, we aim to predict and describe the trajectory of *fluid particles* (§0.2.2) as they travel.

Newton's second law allows us to quantify how the velocity vector of a particle varies with time. If we know all of the forces to which one particle is subjected, we can obtain a streak of velocity vectors $\vec{V}_{\text{particle}} = (u, v, w) = f(x, y, z, t)$ as the particle moves through our area of interest. This is a description of the velocity of *one* particle; we obtain a function of time which depends on where and when the particle started its travel: $\vec{V}_{\text{particle}} = f(x_0, y_0, z_0, t_0, t)$. This is the process used in solid mechanics when we wish to describe the movement of an object, for example, a single satellite in orbit.

This kind of description, however, is poorly suited to the description of fluid flow, for three reasons:

- Firstly, in order to describe a given fluid flow (e.g. air flow around the rear-view mirror standing out of a car), we would need a large number of initial points (x_0, y_0, z_0, t_0), and we would then obtain as many trajectories $\vec{V}_{\text{particle}}$. It then becomes very difficult to describe to study and describe a problem that is local in space (e.g. the wake immediately behind the car mirror), because this requires finding out where the particles of interest originated, and accounting for the trajectories of each of them.

- Secondly, the concept of a "fluid particle" is not well-suited to the drawing of trajectories. Indeed, not only can particles strain indefinitely, but they can also diffuse into the surrounding particles, "blurring" and blending themselves one into another.

- Finally, the velocity and other properties of a given particle very strongly depend on the properties of the surrounding particles. We have to resolve *simultaneously* the movement equations of all of the particles. A space-based description of properties —one in which we describe properties at a chosen fixed point of coordinates $x_{\text{point}}, y_{\text{point}}, z_{\text{point}}, t_{\text{point}}$— is much more useful than a particle-based description which depends on departure points x_0, y_0, z_0, t_0. It is easier to determine the acceleration of a particle together with that of its current neighbors, than together with that of its initial (former) neighbors.

What we are looking for, therefore, is a description of the velocity fields that is expressed in terms of a fixed observation point $\vec{V}_{\text{point}} = (u, v, w) = f(x_{\text{point}}, y_{\text{point}}, z_{\text{point}}, t)$, through which particles of many different origins may be passing. This is termed a *Eulerian* flow description, as opposed to the particle-based *Lagrangian* description. Grouping all of the point velocities in our flow study zone, we will obtain a velocity field $\vec{V}_{\text{point}}$ that is a function of time.

4.2.2 Total time derivative

Let us imagine a canal in which water is flowing at constant and uniform speed $u = U_{\text{canal}}$ (fig. 4.1). The temperature T_{water} of the water is constant (in time), but not uniform (in space). We measure this temperature with a stationary probe, reading $T_{\text{probe}} = T_{\text{water}}$ on the instrument. Even though the temperature T_{water} is constant, when reading the value measured at the probe, temperature will be changing with time:

$$\begin{cases} \frac{\mathrm{d}T_{\text{water}}}{\mathrm{d}t} &= 0 \\ \frac{\mathrm{d}T_{\text{probe}}}{\mathrm{d}t} &= \frac{\mathrm{d}T_{\text{water}}}{\mathrm{d}x}\, u_{\text{water}} \end{cases}$$

In the case where the water temperature is not simply heterogeneous, but also unsteady (varying in time with a rate $\partial T/\partial t$), then that local change in time will also affect the rate which is measured at the probe:

$$\begin{cases} \frac{\mathrm{d}T_{\text{water}}}{\mathrm{d}t} & \neq \quad 0 \\ \frac{\mathrm{d}T_{\text{probe}}}{\mathrm{d}t} & = \quad \frac{\partial T_{\text{water}}}{\partial t}\Big|_{\text{probe}} + \frac{\partial T_{\text{water}}}{\partial x} u_{\text{water}} \end{cases} \tag{4/1}$$

We keep in mind that the rate $\partial T/\partial t$ can itself be a function of time and space; in equation 4/1, it is its value at the position of the probe and at the time of measurement which is taken into account.

This line of thought can be generalized for three dimensions and for any property A of the fluid (including vector properties). The property A of one individual particle can vary as it is moving, so that it has a distribution $A = f(x, y, z, t)$ within the fluid. The time rate change of A measured at a point fixed in space is named the *total time derivative* or simply *total derivative*[1] of A and written $\mathrm{D}A/\mathrm{D}t$:

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{4/2}$$

$$\frac{\mathrm{D}A}{\mathrm{D}t} = \frac{\partial A}{\partial t} + u\frac{\partial A}{\partial x} + v\frac{\partial A}{\partial y} + w\frac{\partial A}{\partial z} \tag{4/3}$$

$$\frac{\mathrm{D}\vec{A}}{\mathrm{D}t} = \frac{\partial \vec{A}}{\partial t} + u\frac{\partial \vec{A}}{\partial x} + v\frac{\partial \vec{A}}{\partial y} + w\frac{\partial \vec{A}}{\partial z} \tag{4/4}$$

$$= \begin{pmatrix} \frac{\partial A_x}{\partial t} + u\frac{\partial A_x}{\partial x} + v\frac{\partial A_x}{\partial y} + w\frac{\partial A_x}{\partial z} \\ \frac{\partial A_y}{\partial t} + u\frac{\partial A_y}{\partial x} + v\frac{\partial A_y}{\partial y} + w\frac{\partial A_y}{\partial z} \\ \frac{\partial A_z}{\partial t} + u\frac{\partial A_z}{\partial x} + v\frac{\partial A_z}{\partial y} + w\frac{\partial A_z}{\partial z} \end{pmatrix} \tag{4/5}$$

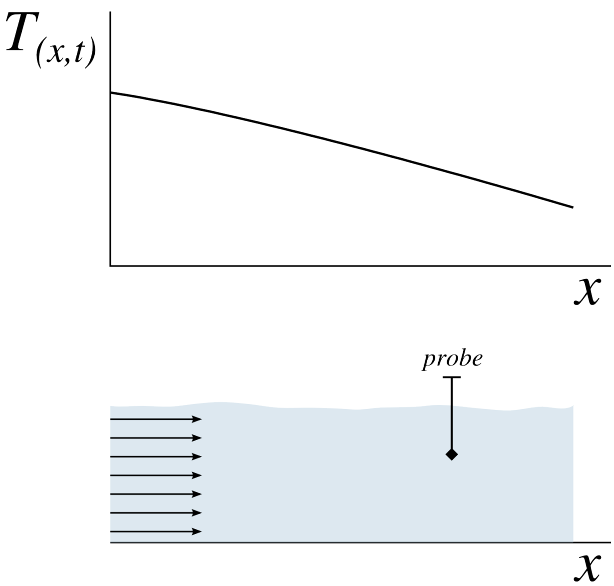


Figure 4.1 – A one-dimensional water flow, for example in a canal. The water has a non-uniform temperature, which, even if it is constant in time, translates in a temperature *rate change in time* at the probe.

[1]Unfortunately this term has many denominations across the litetature, including *advective*, *convective*, *hydrodynamic*, *Lagrangian*, *particle*, *substantial*, *substantive*, or *Stokes derivative*. In this document, the term *total derivative* is used.

In eq. equation 4/3, it is possible simplify the notation of the last three terms. We know the coordinates of the velocity vector $\vec{V}$ (by definition) and of the operator gradient $\vec{\nabla}$ (as defined in eq. 1/8 p. 32) are:

$$\vec{V} \equiv \vec{i}\, u + \vec{j}\, v + \vec{k}\, w$$
$$\vec{\nabla} \equiv \vec{i}\, \frac{\partial}{\partial x} + \vec{j}\, \frac{\partial}{\partial y} + \vec{k}\, \frac{\partial}{\partial z}$$

We thus define the *advective operator*, $(\vec{V} \cdot \vec{\nabla})$:

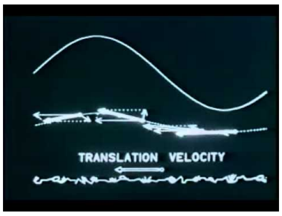
$$\vec{V} \cdot \vec{\nabla} = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \tag{4/6}$$

We can now rewrite eqs. 4/2 and 4/3 in a more concise way:

$$\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + (\vec{V} \cdot \vec{\nabla}) \tag{4/7}$$

$$\frac{\mathrm{D}A}{\mathrm{D}t} = \frac{\partial A}{\partial t} + (\vec{V} \cdot \vec{\nabla})A \tag{4/8}$$

As we explored in equation 4/1, the total derivative of a property can be non-null even if the flow is perfectly steady; similarly, the total derivative of a property can be zero (e.g. when a stationary probe provides a constant measurement) even if the properties of the particles have a non-zero time change (e.g. a fluid with falling temperature, but with a non-uniform temperature).



Video: half-century-old, but timeless didactic exploration of the concept of Lagrangian and Eulerian derivatives, with accompanying notes by Lumley[2]
by the National Committee for Fluid Mechanics Films (NCFMF, 1969[11]) (STYL)
https://youtu.be/mdN8OOkx2ko

The total time derivative is the tool that we were looking for. Now, as we solve for the fluid properties in a stationary reference frame, instead of in the reference frame of a moving particle, we will replace all the time derivatives $\frac{\mathrm{d}}{\mathrm{d}t}$ with *total* derivatives $\frac{\mathrm{D}}{\mathrm{D}t}$. This allows us to compute the change in time of a property *locally*, without the need to track the movement of particles along our field of study. The first and most important such property we are instested in is velocity. Instead of solving for the acceleration of numerous particles ($\mathrm{d}\vec{V}_{\text{particle}}/\mathrm{d}t$), we will focus on calculating the *acceleration field* $\mathrm{D}\vec{V}/\mathrm{D}t$.

4.3 Mass conservation

The first physical principle we wish to express in a derivative manner is the conservation of mass (eq. 0/22 p.19). Let us consider a fluid particle of volume $\mathrm{d}\mathcal{V}$, at a given instant in time (fig. 4.2). We can reproduce our analysis from chapter 3 by quantifying the time change of mass within an arbitrary volume.

In the present case, the control volume is stationary and the particle (our system) is flowing through it. We start with eq. 3/5 p.64:

$$\frac{\mathrm{d}m_{\text{particle}}}{\mathrm{d}t} = 0 = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\mathrm{CV}} \rho\, \mathrm{d}\mathcal{V} + \iint_{\mathrm{CS}} \rho (\vec{V}_{\text{rel}} \cdot \vec{n})\, \mathrm{d}A \tag{4/9}$$

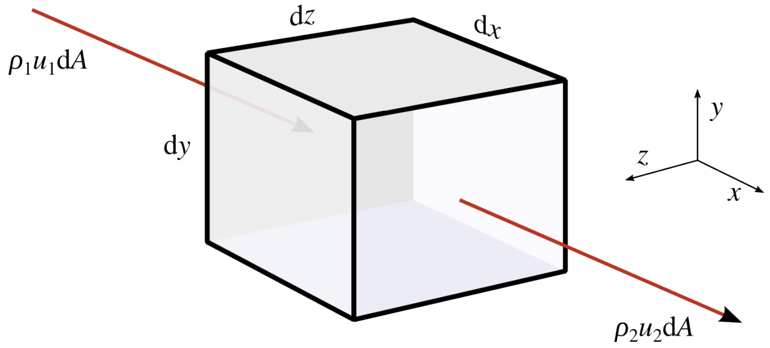


Figure 4.2 – Conservation of mass within a fluid particle. In the x-direction, a mass flow $\dot{m}_1 = \iint \rho_1 u_1 \, \mathrm{d}z \, \mathrm{d}y$ is flowing in, and a mass flow $\dot{m}_2 = \iint \rho_2 u_2 \, \mathrm{d}z \, \mathrm{d}y$ is flowing out. These two flows may not be equal, since mass may also flow in the y- and z-directions.

Figure CC-0 o.c.

The first of these two integrals can be rewritten using the Leibniz integral rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\mathrm{CV}} \rho \, \mathrm{d}\mathcal{V} &= \iiint_{\mathrm{CV}} \frac{\partial \rho}{\partial t} \mathrm{d}\mathcal{V} + \iint_{\mathrm{CS}} \rho V_S \, \mathrm{d}A \\ &= \iiint_{\mathrm{CV}} \frac{\partial \rho}{\partial t} \mathrm{d}\mathcal{V} \end{aligned} \tag{4/10}$$

where V_S is the speed of the control volume wall;
and where the term $\iint_{\mathrm{CS}} \rho V_S \, \mathrm{d}A$ is simply zero because we chose a fixed control volume, such as a fixed computation grid[1].

Now we turn to the second term of equation 4/9, $\iint_{\mathrm{CS}} \rho(\vec{V}_{\mathrm{rel}} \cdot \vec{n}) \, \mathrm{d}A$, which represents the mass flow $\dot{m}_{\mathrm{net}}$ flowing through the control volume.

In the direction x, the mass flow $\dot{m}_{\mathrm{net}\ x}$ flowing through our control volume can be expressed as:

$$\begin{aligned} \dot{m}_{\mathrm{net}\ x} &= \iint_{\mathrm{CS}} -\rho_1 |u_1| \, \mathrm{d}z \, \mathrm{d}y + \iint_{\mathrm{CS}} \rho_2 |u_2| \, \mathrm{d}z \, \mathrm{d}y \\ &= \iint_{\mathrm{CS}} \int \frac{\partial}{\partial x} (\rho u) \, \mathrm{d}x \, \mathrm{d}z \, \mathrm{d}y \\ &= \iiint_{\mathrm{CV}} \frac{\partial}{\partial x} (\rho u) \, \mathrm{d}\mathcal{V} \end{aligned} \tag{4/11}$$

The same applies for directions y and z, so that we can write:

$$\begin{aligned} \iint_{\mathrm{CS}} \rho(\vec{V}_{\mathrm{rel}} \cdot \vec{n}) \, \mathrm{d}A = \dot{m}_{\mathrm{net}} &= \dot{m}_{\mathrm{net}\ x} + \dot{m}_{\mathrm{net}\ y} + \dot{m}_{\mathrm{net}\ z} \\ &= \iiint_{\mathrm{CV}} \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \right] \mathrm{d}\mathcal{V} \\ &= \iiint_{\mathrm{CV}} \vec{\nabla} \cdot (\rho \vec{V}) \, \mathrm{d}\mathcal{V} \end{aligned} \tag{4/12}$$

[1] In a CFD calculation in which the grid is deforming, this term $\iint_{\mathrm{CS}} \rho V_S \, \mathrm{d}A$ will have to be re-introduced in the continuity equation.

Now, with these two equations 4/10 and 4/12, we can come back to equation 4/9, which becomes:

$$\begin{aligned}\frac{\mathrm{d}m_{\text{particle}}}{\mathrm{d}t} = 0 = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathrm{CV}} \rho \, \mathrm{d}\mathcal{V} + \iint_{\mathrm{CS}} \rho(\vec{V}_{\text{rel}} \cdot \vec{n}) \, \mathrm{d}A \\ 0 = \iiint_{\mathrm{CV}} \frac{\partial \rho}{\partial t} \mathrm{d}\mathcal{V} \ + \iiint_{\mathrm{CV}} \vec{\nabla} \cdot (\rho \vec{V}) \ \mathrm{d}\mathcal{V}\end{aligned} \tag{4/13}$$

Since we are only concerned with a very small volume $\mathrm{d}\mathcal{V}$, we drop the integrals, obtaining:

$$0 = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) \tag{4/14}$$

for all flows, with all fluids.

This equation 4/14 is named *continuity equation* and is of crucial importance in fluid mechanics. All fluid flows, in all conditions and at all times, obey this law.

Equation 4/14 is a scalar equation: it gives us no particular information about the orientation of $\vec{V}$ or about its change in time. In itself, this is insufficient to solve the majority of problems in fluid mechanics, and in practice it is used as a kinematic constraint to solutions used to evaluate their physicality or the quality of the approximations made to obtain them.

The most assiduous readers will have no difficulty reading the following equation,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = \frac{\partial \rho}{\partial t} + \vec{V} \cdot \vec{\nabla} \rho + \rho (\vec{\nabla} \cdot \vec{V}) = 0 \tag{4/15}$$

which allows us to re-write equation 4/14 like so:

$$\frac{1}{\rho} \frac{\mathrm{D}\rho}{\mathrm{D}t} + \vec{\nabla} \cdot \vec{V} = 0 \tag{4/16}$$

Therefore, we can see that for an incompressible flow, the divergent of velocity $\vec{\nabla} \cdot \vec{V}$ (sometimes called *volumetric dilatation rate*) is zero:

$$\vec{\nabla} \cdot \vec{V} = 0 \tag{4/17}$$

for any incompressible flow.

In three Cartesian coordinates, this is re-expressed as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4/18}$$

4.4 Change of linear momentum

Now that we have seen what mass conservation tells us about the dynamics of fluids, we turn to the momentum equation, hoping to express the velocity field to the forces that are applied on the fluid.

4.4.1 The Cauchy equation

We start by writing Newton's second law (eq. 0/23 p.19) as it applies to a fluid particle of mass m_{particle}, as shown in fig. 4.3. Fundamentally, the forces on a fluid particle are of only three kinds, namely weight, pressure, and shear:[1]

$$m_{\text{particle}} \frac{\mathrm{d}\vec{V}}{\mathrm{d}t} = \vec{F}_{\text{weight}} + \vec{F}_{\text{net, pressure}} + \vec{F}_{\text{net, shear}} \tag{4/19}$$

We now apply this equation to a stationary cube of infinitesimal volume $\mathrm{d}\mathcal{V}$ which is traversed by fluid. Measuring the velocity from the reference frame of the cube, and thus using a Eulerian description, we obtain:

$$\begin{aligned}\frac{m_{\text{particle}}}{\mathrm{d}\mathcal{V}} \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} &= \frac{m}{\mathrm{d}\mathcal{V}}\vec{g} + \frac{1}{\mathrm{d}\mathcal{V}}\vec{F}_{\text{net, pressure}} + \frac{1}{\mathrm{d}\mathcal{V}}\vec{F}_{\text{net, shear}} \\ \rho\frac{\mathrm{D}\vec{V}}{\mathrm{D}t} &= \rho\vec{g} + \frac{1}{\mathrm{d}\mathcal{V}}\vec{F}_{\text{net, pressure}} + \frac{1}{\mathrm{d}\mathcal{V}}\vec{F}_{\text{net, shear}}\end{aligned} \tag{4/20}$$

Therefore, we will have obtained an expression for the acceleration field $\mathrm{D}\vec{V}/\mathrm{D}t$ if we can find an expression for the net pressure and shear force. Let us quantify those efforts as they apply on each of the faces of the fluid

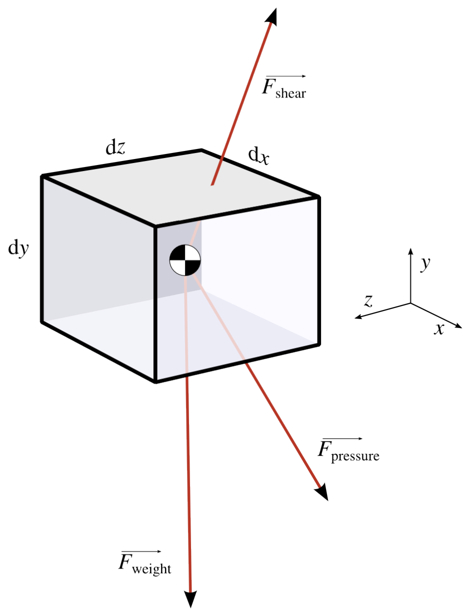


Figure 4.3 – In our study of fluid mechanics, we consider only forces due to gravity, shear, or pressure.

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[1] In some very special and rare applications, electromagnetic forces may also apply.

particle. For pressure, we obtain:

$$\begin{aligned}
F_{\text{net, pressure},x} &= \mathrm{d}y\,\mathrm{d}z\,[p_1 - p_4] \\
&= \mathrm{d}y\,\mathrm{d}z\left[-\frac{\partial p}{\partial x}\,\mathrm{d}x\right]
\end{aligned}$$

$$F_{\text{net, pressure},x} = \mathrm{d}\mathcal{V}\frac{-\partial p}{\partial x} \tag{4/21}$$

$$F_{\text{net, pressure},y} = \mathrm{d}\mathcal{V}\frac{-\partial p}{\partial y} \tag{4/22}$$

$$F_{\text{net, pressure},z} = \mathrm{d}\mathcal{V}\frac{-\partial p}{\partial z} \tag{4/23}$$

This pattern, we had seen in chapter 1 (with eq.1/8 p.32), can more elegantly be expressed with the gradient operator (see also Appendix A2 p.217), allowing us to write:

$$\frac{1}{\mathrm{d}\mathcal{V}}\vec{F}_{\text{net, pressure}} = -\vec{\nabla}p \tag{4/24}$$

As for shear, we can put our work from chapter 2 to good use now. There, we had seen that the effect of shear on a volume had eighteen components, the careful examination of which allowed us to quantify the net *force* due to shear (which has only three components). In the x-direction, for example, we had seen (with eq. 2/8 p.49, as illustrated again in fig. 4.4) that the net shear force in the x-direction $\vec{F}_{\text{shear}x}$ was the result of shear in the x-direction on each of the six faces:

$$\begin{aligned}
\vec{F}_{\text{shear }x} = \;& \mathrm{d}x\,\mathrm{d}y(\vec{\tau}_{zx\ 3} - \vec{\tau}_{zx\ 6}) \\
&+ \mathrm{d}x\,\mathrm{d}z(\vec{\tau}_{yx\ 2} - \vec{\tau}_{yx\ 5}) \\
&+ \mathrm{d}z\,\mathrm{d}y(\vec{\tau}_{xx\ 1} - \vec{\tau}_{xx\ 4})
\end{aligned}$$

This, in turn, was more elegantly expressed using the divergent operator (see also Appendix A2 p.217) as equation 2/13:

$$\vec{F}_{\text{shear }x} = \mathrm{d}\mathcal{V}\left(\frac{\partial\vec{\tau}_{zx}}{\partial z} + \frac{\partial\vec{\tau}_{yx}}{\partial y} + \frac{\partial\vec{\tau}_{xx}}{\partial x}\right) \tag{4/25}$$

$$= \mathrm{d}\mathcal{V}\;\vec{\nabla}\cdot\vec{\tau}_{ix} \tag{4/26}$$

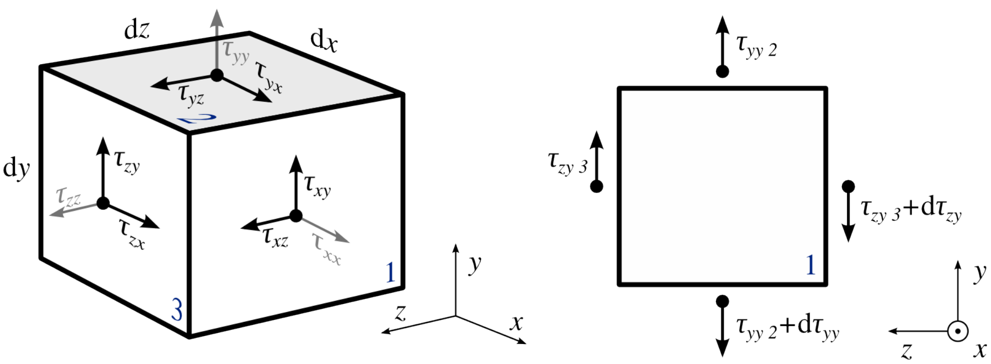

Figure 4.4 – Shear efforts on a cubic fluid particle (already represented in fig. 2.1 p.48).

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Finally, we had assembled the components in all three directions into a single expression for the net shear force with eq. 2/14 p.50:

$$\vec{F}_{\text{shear}} = \mathrm{d}\mathcal{V}\ \vec{\nabla} \cdot \vec{\tau}_{ij} \tag{4/27}$$

Now, we can put together our findings, eq. 4/24 for the force due to pressure and eq. 4/27 for the force due to shear, back into equation 4/20, we obtain the *Cauchy equation*:

$$\rho \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \rho\vec{g} - \vec{\nabla}p + \vec{\nabla} \cdot \vec{\tau}_{ij} \tag{4/28}$$

for all flows, with all fluids.

The Cauchy equation is an expression of Newton's second law applied to a fluid particle. It expresses the time change of the velocity vector measured at a fixed point (the acceleration field $\mathrm{D}\vec{V}/\mathrm{D}t$) as a function of gravity, pressure and shear effects. This is quite a breakthrough. Within the chaos of an arbitrary flow, in which fluid particles are shoved, pressurized, squeezed, and distorted, we know precisely what we need to look for in order to quantify the time-change of velocity: gravity, the gradient of pressure, and the divergent of shear.

Nevertheless, while it is an excellent start, this equation isn't detailed enough for us. In our search for the velocity field $\vec{V}$, the changes in time and space of the shear tensor $\vec{\tau}_{ij}$ and pressure p are unknowns. Ideally, those two terms should be expressed solely as a function of the flow's other properties. Obtaining such an expression is what Claude-Louis Navier and Gabriel Stokes set themselves to in the 19th century: we follow their footsteps in the next paragraphs.

4.4.2 The Navier-Stokes equation for incompressible flow

The Navier-Stokes equation is the Cauchy equation (eq. 4/28) applied to Newtonian fluids. In Newtonian fluids, which we encountered in chapter 2 (§2.3.3 p.52), shear efforts are simply proportional to the rate of strain; thus, the shear component of eq. 4/28 can be re-expressed usefully.

We had seen with eq. 2/16 p.51 that the norm $||\vec{\tau}_{ij}||$ of shear component in direction j along a surface perpendicular to i can be expressed as:

$$||\vec{\tau}_{ij}|| = \mu \frac{\partial u_j}{\partial i}$$

Thus, in a Newtonian fluid, on each of the six faces of the infinitesimal cube from fig. 4.4, the shear in x-direction is proportional to the partial derivative of u (the component of velocity in the x-direction) with respect to the direction perpendicular to each face:

$$\begin{aligned}
\vec{\tau}_{xx1} &= \mu \left.\frac{\partial u}{\partial x}\right|_1 \vec{i} & \qquad \vec{\tau}_{xx4} &= -\mu \left.\frac{\partial u}{\partial x}\right|_4 \vec{i} \\
\vec{\tau}_{yx2} &= \mu \left.\frac{\partial u}{\partial y}\right|_2 \vec{i} & \qquad \vec{\tau}_{yx5} &= -\mu \left.\frac{\partial u}{\partial y}\right|_5 \vec{i} \\
\vec{\tau}_{zx3} &= \mu \left.\frac{\partial u}{\partial z}\right|_3 \vec{i} & \qquad \vec{\tau}_{zx6} &= -\mu \left.\frac{\partial u}{\partial z}\right|_6 \vec{i}
\end{aligned}$$

These are the six constituents of the x-direction shear tensor $\vec{\tau}_{ix}$. Their net effect is a vector expressed as the divergent $\vec{\nabla} \cdot \vec{\tau}_{ix}$, which, in incompressible flow[1], is expressed as:

$$\begin{aligned}
\vec{\nabla} \cdot \vec{\tau}_{ix} &= \frac{\partial \vec{\tau}_{xx}}{\partial x} + \frac{\partial \vec{\tau}_{yx}}{\partial y} + \frac{\partial \vec{\tau}_{zx}}{\partial z} \\
&= \frac{\partial \left(\mu \frac{\partial u}{\partial x} \vec{i} \right)}{\partial x} + \frac{\partial \left(\mu \frac{\partial u}{\partial y} \vec{i} \right)}{\partial y} + \frac{\partial \left(\mu \frac{\partial u}{\partial z} \vec{i} \right)}{\partial z} \\
&= \mu \frac{\partial \left(\frac{\partial u}{\partial x} \right)}{\partial x} \vec{i} + \mu \frac{\partial \left(\frac{\partial u}{\partial y} \right)}{\partial y} \vec{i} + \mu \frac{\partial \left(\frac{\partial u}{\partial z} \right)}{\partial z} \vec{i} \\
&= \mu \left(\frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial y)^2} + \frac{\partial^2 u}{(\partial z)^2} \right) \vec{i} \qquad (4/29)
\end{aligned}$$

Using the *Laplacian* operator (see also Appendix A2 p.217) to represent the spatial variation of the spatial variation of an object:

$$\vec{\nabla}^2 \equiv \vec{\nabla} \cdot \vec{\nabla} \qquad (4/30)$$

$$\vec{\nabla}^2 A \equiv \vec{\nabla} \cdot \vec{\nabla} A \qquad (4/31)$$

$$\vec{\nabla}^2 \vec{A} \equiv \begin{pmatrix} \vec{\nabla}^2 A_x \\ \vec{\nabla}^2 A_y \\ \vec{\nabla}^2 A_z \end{pmatrix} = \begin{pmatrix} \vec{\nabla} \cdot \vec{\nabla} A_x \\ \vec{\nabla} \cdot \vec{\nabla} A_y \\ \vec{\nabla} \cdot \vec{\nabla} A_z \end{pmatrix} \qquad (4/32)$$

we can re-write eq. 4/29 more elegantly and generalize to three dimensions:

$$\vec{\nabla} \cdot \vec{\tau}_{ix} = \mu \vec{\nabla}^2 u \, \vec{i} = \mu \vec{\nabla}^2 \vec{u} \qquad (4/33)$$

$$\vec{\nabla} \cdot \vec{\tau}_{iy} = \mu \vec{\nabla}^2 v \, \vec{j} = \mu \vec{\nabla}^2 \vec{v} \qquad (4/34)$$

$$\vec{\nabla} \cdot \vec{\tau}_{iz} = \mu \vec{\nabla}^2 w \, \vec{k} = \mu \vec{\nabla}^2 \vec{w} \qquad (4/35)$$

The three last equations are grouped together simply as

$$\vec{\nabla} \cdot \vec{\tau}_{ij} = \mu \vec{\nabla}^2 \vec{V} \qquad (4/36)$$

With this new expression, we can come back to the Cauchy equation (eq. 4/28), in which we can replace the shear term with eq. 4/36. This produces the glorious *Navier-Stokes equation for incompressible flow*:

$$\rho \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} = \rho \vec{g} - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{V} \qquad (4/37)$$

for all incompressible flows of a Newtonian fluid.

This formidable equation describes the property fields of all incompressible flows of Newtonian fluids. It expresses the acceleration field (left-hand side) as the sum of three contributions (right-hand side): those of gravity, gradient of pressure, and divergent of shear. The solutions we look for in equation 4/37 are the velocity (vector) field $\vec{V} = (u, v, w) = f_1(x, y, z, t)$ and the pressure field $p = f_2(x, y, z, t)$, given a set of constraints to represent the problem at hand. These constraints, called *boundary conditions*, may be expressed in terms of

[1]When the flow becomes compressible, normal stresses τ_{xx} will see an additional term related to the change in density within the particle. An additional term, $+\frac{1}{3}\mu\vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right)$, would appear on the right side of eq. 4/37. This is well outside of the scope of this course.

velocities (e.g. the presence of a fixed solid body is expressed as a region for which $\vec{V} = \vec{0}$) or pressure (e.g. a discharge into an open atmosphere may be expressed as a region of known constant pressure).

The combination of equations 4/37 and 4/14 (continuity & Navier-Stokes) may not be enough to predict the solution of a given problem, most especially if large energy transfers take place within the flow. In that case, the addition of an energy equation and an expression of the second law of thermodynamics, both in a form suitable for derivative analysis, may be needed to provide as many equations as there are unknowns.

Though it is without doubt charming, equation 4/37 should be remembered for what it is really: a three-dimensional system of coupled equations. In Cartesian coordinates this complexity is more apparent:

$$\rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right] = \rho g_x - \frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial y)^2} + \frac{\partial^2 u}{(\partial z)^2} \right] \tag{4/38}$$

$$\rho \left[\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right] = \rho g_y - \frac{\partial p}{\partial y} + \mu \left[\frac{\partial^2 v}{(\partial x)^2} + \frac{\partial^2 v}{(\partial y)^2} + \frac{\partial^2 v}{(\partial z)^2} \right] \tag{4/39}$$

$$\rho \left[\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right] = \rho g_z - \frac{\partial p}{\partial z} + \mu \left[\frac{\partial^2 w}{(\partial x)^2} + \frac{\partial^2 w}{(\partial y)^2} + \frac{\partial^2 w}{(\partial z)^2} \right] \tag{4/40}$$

Today indeed, 150 years after it was first written, no general expression has been found for velocity or pressure fields that would solve this vector equation in the general case. Nevertheless, in this course we will use it directly:

- to understand and quantify the importance of key fluid flow parameters, in chapter 6;
- to find analytical solutions to flows in a few selected cases, in the other remaining chapters.

After this course, the reader might also engage into *computational fluid dynamics* (CFD) a discipline entirely architectured around this equation, and to which it purposes to find solutions as fields of discrete values.

As a finishing remark, we note that when the flow is strictly two-dimensional, the Navier-Stokes equation is considerably simplified, shrinking down to the system:

$$\rho \left[\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = \rho g_x - \frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial y)^2} \right] \tag{4/41}$$

$$\rho \left[\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right] = \rho g_y - \frac{\partial p}{\partial y} + \mu \left[\frac{\partial^2 v}{(\partial x)^2} + \frac{\partial^2 v}{(\partial y)^2} \right] \tag{4/42}$$

Unfortunately, this simplification prevents us from accounting for turbulence, which is a strongly three-dimensional phenomenon.

4.4.3 The Bernoulli equation

We had made clear in chapter 3 (§3.7) that the Bernoulli equation was extraordinarily limited in scope, so much that starting from the more general integral expressions for energy or momentum conservation was much safer in practice. As a reminder of this fact, and as an illustration of the bridges that can be built between integral and derivative analysis, it can be instructive to derive the Bernoulli equation directly from the Navier-Stokes equation.

Let us start by following a particle along its path in an arbitrary flow, as displayed in fig. 4.5. The particle path is known (condition 5 in §3.7), but its speed V is not.

We are now going to project every component of the Navier-Stokes equation (eq. 4/37) onto an infinitesimal portion of trajectory $\mathrm{d}\vec{s}$. Once all terms have been projected, the Navier-Stokes equation becomes simply a scalar equation:

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho (\vec{V} \cdot \vec{\nabla}) \vec{V} = \rho \vec{g} - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{V}$$

$$\rho \frac{\partial \vec{V}}{\partial t} \cdot \mathrm{d}\vec{s} + \rho (\vec{V} \cdot \vec{\nabla}) \vec{V} \cdot \mathrm{d}\vec{s} = \rho \vec{g} \cdot \mathrm{d}\vec{s} - \vec{\nabla} p \cdot \mathrm{d}\vec{s} + \mu \vec{\nabla}^2 \vec{V} \cdot \mathrm{d}\vec{s}$$

Because the velocity vector $\vec{V}$ of the particle, by definition, is always aligned with the path, its projection is always equal to its norm: $\vec{V} \cdot \mathrm{d}\vec{s} = V \, \mathrm{d}s$. Also, the downward gravity g and the upward altitude z have opposite signs, so that $\vec{g} \cdot \mathrm{d}\vec{s} = -g \, \mathrm{d}z$; we thus obtain:

$$\rho \frac{\partial V}{\partial t} \mathrm{d}s + \rho \frac{\mathrm{d}V}{\mathrm{d}s} V \, \mathrm{d}s = -\rho g \, \mathrm{d}z - \frac{\mathrm{d}p}{\mathrm{d}s} \mathrm{d}s + \mu \vec{\nabla}^2 \vec{V} \cdot \mathrm{d}\vec{s}$$

When we restrict ourselves to steady flow (condition 1 in §3.7), the first left-hand term vanishes. Neglecting losses to friction (condition 4) alleviates us from the last right-hand term, and we obtain:

$$\rho \frac{\mathrm{d}V}{\mathrm{d}s} V \, \mathrm{d}s = -\rho g \, \mathrm{d}z - \frac{\mathrm{d}p}{\mathrm{d}s} \mathrm{d}s$$
$$\rho V \, \mathrm{d}V = -\rho g \, \mathrm{d}z - \mathrm{d}p$$

This equation can then be integrated from point 1 to point 2 along the pathline:

$$\rho \int_1^2 V \, \mathrm{d}V = -\int_1^2 \rho g \, \mathrm{d}z - \int_1^2 \mathrm{d}p$$

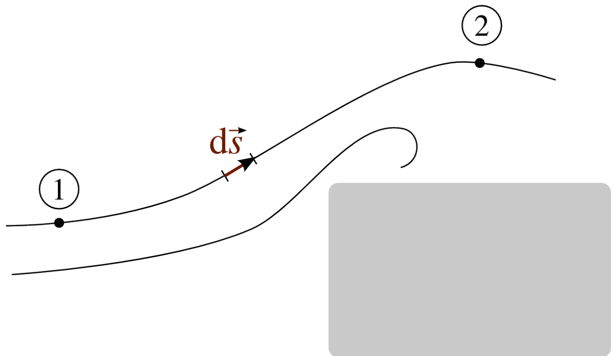

Figure 4.5 – Different pathlines in an arbitrary flow. We follow one particle as it travels from point 1 to point 2. An infinitesimal path segment is named $\mathrm{d}\vec{s}$.

Figure CC-0 o.c.

When no work or heat transfer occurs (condition 3) and the flow remains incompressible (condition 2), the density ρ remains constant, so that we indeed have returned to eq. 3/18:

$$\Delta\left(\frac{1}{2}V^2\right) + g\Delta z + \frac{1}{\rho}\Delta p = 0 \tag{4/43}$$

Thus, we can see that if we follow a particle along its path, in a steady, incompressible, frictionless flow with no heat or work transfer, its change in kinetic energy is due only to the result of gravity and pressure, in accordance with the Navier-Stokes equation.

4.5 Change of angular momentum

There is no differential equation for conservation of angular momentum in fluid flow. Indeed, as the size of a considered particle tends to zero, it loses its ability to possess (or exchange) angular momentum about its center of gravity. Infinitesimal fluid particles are not able to rotate about themselves, only about an external point. So, in this chapter, no analysis is carried out using the concept of angular momentum.

4.6 Energy conservation and increase in entropy

This topic is well covered in Anderson [7]

The last two key principles used in fluid flow analysis can be written together in a differential equation similar to the Navier-Stokes equation.

Once again, we start from the analysis of transfers on an infinitesimal control volume. We are going to relate three energy terms in the following form, naming them A, B and C for clarity:

$$\begin{array}{ccccc} \text{the rate of change} & & \text{the net flux} & & \text{the rate of work done} \\ \text{of energy inside} & = & \text{of heat into} & + & \text{on the element due to} \\ \text{the fluid element} & & \text{the element} & & \text{body and surface forces} \\ A & = & B & + & C \end{array} \tag{4/44}$$

Let us first evaluate term C. The rate of work done on the particle is the dot product of its velocity $\vec{V}$ and the net force $\vec{F}_{\text{net}}$ applying to it:

$$C = \vec{V} \cdot \left(\frac{\vec{F}_{\text{weight}}}{\mathrm{d}\mathcal{V}} + \frac{\vec{F}_{\text{net, pressure}}}{\mathrm{d}\mathcal{V}} + \frac{\vec{F}_{\text{net, shear}}}{\mathrm{d}\mathcal{V}} \right) \mathrm{d}\mathcal{V}$$
$$C = \vec{V} \cdot \left(\rho\vec{g} - \vec{\nabla}p + \vec{\nabla} \cdot \vec{\tau}_{ij} \right) \mathrm{d}\mathcal{V}$$

and for the case of a Newtonian fluid in an incompressible flow, this expression can be re-written as:

$$C = \vec{V} \cdot \left(\rho\vec{g} - \vec{\nabla}p + \mu\vec{\nabla}^2\vec{V} \right) \mathrm{d}\mathcal{V} \tag{4/45}$$

We now turn to term B, the net flux of heat into the element. We attribute this flux to two contributions, the first named $\dot{Q}_{\text{radiation}}$ from the emission or absorption of radiation, and the second, named $\dot{Q}_{\text{conduction}}$ to thermal conduction through the faces of the element. We make no attempt to quantify $\dot{Q}_{\text{radiation}}$, simply stating that

$$\dot{Q}_{\text{radiation}} = \rho \dot{q}_{\text{radiation}} \, \mathrm{d}\mathcal{V} \tag{4/46}$$

in which $\dot{q}_{\text{radiation}}$ is the local power per unit mass (in $\mathrm{W\,kg^{-1}}$) transfered to the element, to be determined from the boundary conditions and flow temperature distribution.

In the x-direction, thermal conduction through the faces of the element causes a net flow of heat $\dot{Q}_{\text{conduction},x}$ expressed as a function of the power per area q (in $\mathrm{W\,m^{-2}}$) through each of the two faces perpendicular to x:

$$\begin{aligned}
\dot{Q}_{\text{conduction},x} &= \left[q_x - \left(q_x + \frac{\partial q_x}{\partial x}\mathrm{d}x\right)\right] \mathrm{d}y \, \mathrm{d}z \\
&= -\frac{\partial q_x}{\partial x} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z && (4/47)\\
&= -\frac{\partial q_x}{\partial x} \mathrm{d}\mathcal{V} && (4/48)
\end{aligned}$$

Summing contributions from all three directions, we obtain:

$$\begin{aligned}
\dot{Q}_{\text{conduction}} &= \dot{Q}_{\text{conduction},x} + \dot{Q}_{\text{conduction},y} + \dot{Q}_{\text{conduction},z} \\
&= -\left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right] \mathrm{d}\mathcal{V} && (4/49)
\end{aligned}$$

So we finally obtain an expression for B:

$$\begin{aligned}
B &= \dot{Q}_{\text{radiation}} + \dot{Q}_{\text{conduction}} \\
&= \left(\rho \dot{q}_{\text{radiation}} - \left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right]\right) \mathrm{d}\mathcal{V} && (4/50)
\end{aligned}$$

Finally, term A, the rate of change of energy inside the fluid element, can be expressed as a function of the specific kinetic energy e_k and specific internal energy i:

$$A = \rho \frac{\mathrm{D}}{\mathrm{D}t}\left(i + \frac{1}{2}V^2\right) \mathrm{d}\mathcal{V} \tag{4/51}$$

We are therefore able to relate the properties of a fluid particle to the principle of energy conservation as follows:

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(i + \frac{1}{2}V^2\right) = \left(\rho \dot{q}_{\text{radiation}} - \left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right]\right) + \vec{V} \cdot \left(\rho \vec{g} - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{V}\right) \tag{4/52}$$

This is a scalar equation — it only has one dimension, and involves the length of the velocity vector, $V \equiv \left[u^2 + v^2 + w^2\right]^{\frac{1}{2}}$.

In an interesting hack, we are able to incorporate an expression for the *second* principle of thermodynamics in this equation simply by expressing the fluxes q_i as a function of the temperature gradients. Indeed, expressing a heat flux as the result of a difference in temperature according to the Fourier law,

$$q_i = -k \frac{\partial T}{\partial i} \tag{4/53}$$

we constrain the direction of the heat fluxes and thus ensure that all dissipative terms resulting in temperature increases cannot be fed back into other energy terms, thus increasing the overall entropy.

Equation 4/53 inserted into eq. 4/52 yields:

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(i + \frac{1}{2}V^2\right) = \left(\rho \dot{q}_{\text{radiation}} + k\left[\frac{\partial^2 T}{(\partial x)^2} + \frac{\partial^2 T}{(\partial y)^2} + \frac{\partial^2 T}{(\partial z)^2}\right]\right) + \vec{V} \cdot \left(\rho \vec{g} - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{V}\right)$$

Making use of a Laplacian operator, we come to:

$$\boxed{\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(i + \frac{1}{2}V^2\right) = \left(\rho \dot{q}_{\text{radiation}} + k\vec{\nabla}^2 T\right) + \vec{V} \cdot \left(\rho \vec{g} - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{V}\right) \qquad (4/54)}$$

This equation has several shortcomings, most importantly because the term $\dot{q}_{\text{radiation}}$ is not expressed in terms of fluid properties, and because μ is typically not independent of the temperature T. Nevertheless, it in principle brings closure to our system of continuity and momentum equations, and these two influences may be either neglected, or modeled numerically.

4.7 CFD: the Navier-Stokes equations in practice

This topic is well covered in Versteeg & Malalasekera [10]

4.7.1 Principle

In our analysis of fluid flow from a derivative perspective, our five physical principles from §0.7 have been condensed into three equations (often loosely referred together to as the *Navier-Stokes equations*). Out of these, the first two, for conservation of mass (4/17) and linear momentum (4/37) in incompressible flows, are often enough to characterize most free flows, and should in principle be enough to find the primary unknown, which is the velocity field $\vec{V}$:

$$\begin{aligned} 0 &= \vec{\nabla} \cdot \vec{V} \\ \rho \frac{\mathrm{D}\vec{V}}{\mathrm{D}t} &= \rho \vec{g} - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{V} \end{aligned}$$

We know of many individual analytical solutions to this mathematical problem: they apply to simple cases, and we shall describe several such flows in the upcoming chapters. However, we do not have one *general* solution: one that would encompass all of them. For example, in solid mechanics we have long understood that *all* pure free fall movements can be described with the solution $x = x_0 + u_0 t$ and $y = y_0 + v_0 t + \frac{1}{2} g t^2$, regardless of the particularities of each fall. In fluid mechanics, even though our analysis was carried out in the same manner, we have yet to find such one general solution — or even to prove that one exists at all.

It is therefore tempting to attack the above pair of equations from the numerical side, with a computer algorithm. If one discretizes space and time in small increments δx, δy, δz and δt, we could re-express the x-component of eq. 4/37 as:

$$\rho\left[\frac{\delta u|_t}{\delta t}+u\frac{\delta u|_x}{\delta x}+v\frac{\delta u|_y}{\delta y}+w\frac{\delta u|_z}{\delta z}\right]=\rho g_x-\frac{\delta p|_x}{\delta x}+\mu\left[\frac{\delta\left.\frac{\delta u|_x}{\delta x}\right|_x}{\delta x}+\frac{\delta\left.\frac{\delta u|_y}{\delta y}\right|_y}{\delta y}+\frac{\delta\left.\frac{\delta u|_z}{\delta z}\right|_z}{\delta z}\right] \tag{4/55}$$

If we start with a *known* (perhaps guessed) initial field for velocity and pressure, this equation 4/55 allows us to isolate and solve for $\delta u|_t$, and therefore predict what the u velocity field would look like after a time increment δt. The same can be done in the y- and z-directions. Repeating the process, we then proceed to the next time step and so on, *marching in time*, obtaining at every new time step the value of u, v and w at each position within our computation grid. This is the fundamental working principle for computational fluid dynamics (CFD) today.

4.7.2 Two problems with CFD

There are two flaws with the process described above.

The first flaw is easy to identify: while we may be able to *start* the calculation with a known pressure field, we do not have a mean to predict p (and thus $\delta p/\delta x$ in eq. 4/55) at the next position in time ($t + \delta t$).
This is not a surprise. Going back to the Cauchy equation (eq. 4/28), we notice that we took care of the divergent of shear $\vec{\nabla} \cdot \vec{\tau}_{ij}$, but not of the gradient of pressure $\vec{\nabla} p$. We have neither an expression $p = f(\vec{V})$, nor an equation that would give us a value for $\partial p/\partial t$. CFD software packages deal with this problem in the most uncomfortable way: by guessing a pressure field $p_{(t+\delta t)}$, calculating the new velocity field $\vec{V}_{(t+\delta t)}$, and then correcting and re-iterating with the help of the continuity equation, until continuity errors are acceptably small. This scheme and its implementation are a great source of anguish amongst numerical fluid dynamicists.

The second flaw appears once we consider the effect of grid coarseness. Every decrease in the size of the grid cell and in the length of the time step increases the total number of equations to be solved by the algorithm. Halving each of δx, δy, δz and δt multiplies the total number of equations by 16, so that soon enough the experimenter will wish to know what maximum (coarsest) grid size is appropriate or tolerable. Furthermore, in many practical cases, we may not even be interested in an exhaustive description of the velocity field, and just wish to obtain a general, coarse description of the fluid flow.

Using a coarse grid and large time step, however, prevents us from resolving small variations in velocity: movements which are so small they fit in between grid points, or so short they occur in between time steps. Let us decompose the velocity field into two components: one is the *average* flow ($\overline{u}, \overline{v}, \overline{w}$), and the other the *instantaneous fluctuation* flow (u', v', w'), too fine to be captured by our grid:

$$u_i \equiv \overline{u}_i + u'_i \tag{4/56}$$

$$\overline{u'_i} \equiv 0 \tag{4/57}$$

With the use of definition 4/56, we re-formulate equation 4/38 as follows:

$$\rho\left[\frac{\partial(\overline{u}+u')}{\partial t}+(\overline{u}+u')\frac{\partial(\overline{u}+u')}{\partial x}+(\overline{v}+v')\frac{\partial(\overline{u}+u')}{\partial y}+(\overline{w}+w')\frac{\partial(\overline{u}+u')}{\partial z}\right]$$
$$=\rho g_x-\frac{\partial(\overline{p}+p')}{\partial x}+\mu\left[\frac{\partial^2(\overline{u}+u')}{(\partial x)^2}+\frac{\partial^2(\overline{u}+u')}{(\partial y)^2}+\frac{\partial^2(\overline{u}+u')}{(\partial z)^2}\right]$$

Taking the *average* of this equation —thus expressing the dynamics of the flow as we calculate them with a finite, coarse grid— yields, after some intimidating but easily conquerable algebra:

$$\rho\left[\frac{\partial\overline{u}}{\partial t}+\overline{u}\frac{\partial\overline{u}}{\partial x}+\overline{v}\frac{\partial\overline{u}}{\partial y}+\overline{w}\frac{\partial\overline{u}}{\partial z}\right]+\rho\left[\overline{u'\frac{\partial u'}{\partial x}}+\overline{v'\frac{\partial u'}{\partial y}}+\overline{w'\frac{\partial u'}{\partial z}}\right]$$
$$=\rho g_x-\frac{\partial\overline{p}}{\partial x}+\mu\left[\frac{\partial^2\overline{u}}{(\partial x)^2}+\frac{\partial^2\overline{u}}{(\partial y)^2}+\frac{\partial^2\overline{u}}{(\partial z)^2}\right] \tag{4/58}$$

Equation 4/58 is the x-component of the *Reynolds-averaged Navier-Stokes equation* (RANS). It shows that when one observes the flow in terms of the sum of an average and an instantaneous component, the dynamics cannot be expressed solely according to the average component. Comparing eqs. 4/58 and 4/38 we find that an additional term has appeared on the left side. This term, called the *Reynolds stress*, is often re-written as $\rho\ \partial\overline{u_i'u_j'}/\partial j$; it is *not zero* since we observe in practice that the instantaneous fluctuations of velocity are strongly correlated.

The difference between eqs. 4/58 and 4/38 can perhaps be expressed differently: the time-average of a real flow cannot be calculated by solving for the time-average velocities. Or, more bluntly: *the average of the solution cannot be obtained with only the average of the flow.* This is a tremendous burden in computational fluid dynamics, where limits on the available computational power prevent us in practice from solving for these fluctuations. In the overwhelming majority of computations, the Reynolds stress has to be approximated in bulk with schemes named *turbulence models*.

Diving into the intricacies of CFD is beyond the scope of our study. Nevertheless, the remarks in this last section should hopefully hint at the fact that an understanding of the *mathematical nature* of the differential conservation equations is of great practical importance in fluid dynamics. It is for that reason that we shall dedicate the exercises of this chapter solely to playing with the mathematics of our two main equations.