Fluid Mechanics
Chapter 3 – Analysis of existing flows

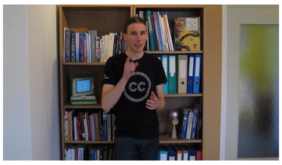
last edited September 23, 2017

These lecture notes are based on textbooks by White [13], Çengel & al.[16], and Munson & al.[18].

3.1 Motivation

Our objective for this chapter is to answer the question "what is the *net* effect of a given fluid flow through a given volume?".

Here, we develop a mass, momentum and energy accounting methodology to analyze the flow of continuous medium. This method is not powerful enough to allow us to describe extensively the nature of fluid flow around bodies; nevertheless, it is extremely useful to quantify forces, moments, and energy transfers associated with fluid flow.



Video: pre-lecture briefing for this chapter, part 1/2

https://youtu.be/1LXlFVtPoCY

3.2 The Reynolds transport theorem

3.2.1 Control volume

Let us begin by describing the flow which interests us a a generic velocity field $\vec{V} = (u, v, w)$ which is a function of space and time ($\vec{V} = f(x, y, z, t)$).

Within this flow, we are interested in an arbitrary volume named *control volume* (CV) which is free to move and change shape (fig. 3.1). We are going to measure the properties of the fluid at the borders of this volume, which we call the *control surface* (CS), in order to compute the net effect of the flow through the volume.

At a given time, the control volume contains a certain amount of mass which we call the *system* (sys). Thus the system is a fixed amount of mass transiting through the control volume at the time of our study, and its properties (volume, pressure, velocity etc.) may change in the process.

All along the chapter, we are focusing on the question: based on measured fluid properties at some point in space and time (the properties at the control surface), how can we quantify what is happening to the system (the mass inside the control volume)?

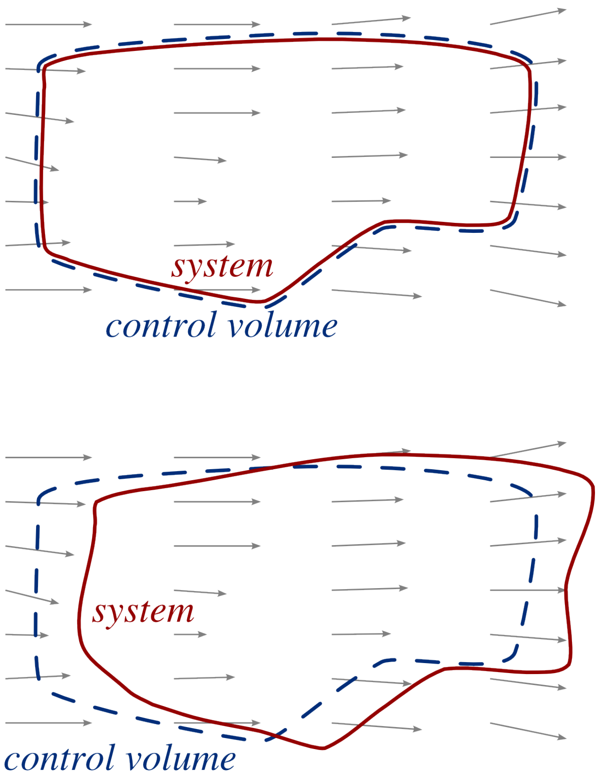


Figure 3.1 – A control volume within a flow. The *system* is the amount of mass included within the control volume at a given time. At a later time, it may have left the control volume, and its shape and properties may have changed. The control volume may also change shape with time, although this is not represented here.

Figure CC-0 o.c.

3.2.2 Rate of change of an additive property

In order to proceed with our calculations, we need a robust accounting methodology. We start with a "dummy" fluid property B, which we will later replace with physical variables of interest.
Let us thus consider an arbitrary additive property B of the fluid. By the term *additive* property, we mean that the total *amount of property* is divided if the fluid is divided. For instance, this is true of mass, volume, energy, entropy, but not pressure or temperature.
The *specific* (i.e. per unit mass) value of B is designated $b \equiv B/m$.

We now wish to compute the variation of a system's property B based on measurements made at the borders of the control volume. We will achieve this with an equation containing three terms:

- The time variation of the quantity B within the system is measured with the term $\frac{\mathrm{d}B_{\text{sys}}}{\mathrm{d}t}$. This may represent, for example, the rate of change of the fluid's internal energy as it travels through a jet engine.

- Within the control volume, the enclosed quantity B_{CV} can vary by accumulation (for example, mass may be increasing in an air tank fed with compressed air): we measure this with the term $\frac{\mathrm{d}B_{\text{CV}}}{\mathrm{d}t}$.

- Finally, a mass flux may be flowing through the boundaries of the control volume, carrying with it some amount of B every second: we name that net flow out of the system $\dot{B}_{\text{net}} \equiv \dot{B}_{\text{out}} - \dot{B}_{\text{in}}$.

We can now link these three terms with the simple equation:

$$\underbrace{\frac{\mathrm{d}B_{\text{sys}}}{\mathrm{d}t}}_{\substack{\text{the rate of change}\\ \text{of } B \text{ for the system}}} = \underbrace{\frac{\mathrm{d}B_{\text{CV}}}{\mathrm{d}t}}_{\substack{\text{the rate of change}\\ \text{of } B \text{ within the}\\ \text{control volume}}} + \underbrace{\dot{B}_{\text{net}}}_{\substack{\text{the net flow of } B\\ \text{through the boundaries}\\ \text{of the control volume}}} \tag{3/1}$$

Since B may not be uniformly distributed within the control volume, we like to express the term $\frac{\mathrm{d}B_{\text{CV}}}{\mathrm{d}t}$ as the integral of the volume density $\frac{B}{\mathcal{V}}$ with respect to volume:

$$\frac{\mathrm{d}B_{\text{CV}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \frac{B}{\mathcal{V}} \, \mathrm{d}\mathcal{V} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho b \, \mathrm{d}\mathcal{V} \tag{3/2}$$

Obtaining a value for this integral may be difficult, especially if the volume of the control volume CV is itself a function of time.

The term $\dot{B}_{\text{net}}$ can be evaluated by quantifying, for each area element $\mathrm{d}A$ of the control volume's surface, the surface flow rate $\rho b V_{\perp}$ of property B that flows through it (fig. 3.2). The integral over the entire control volume surface CS of this term is:

$$\dot{B}_{\text{net}} = \iint_{\text{CS}} \rho b V_{\perp} \, \mathrm{d}A = \iint_{\text{CS}} \rho b \, (\vec{V}_{\text{rel}} \cdot \vec{n}) \, \mathrm{d}A \tag{3/3}$$

where flows and velocities are positive outwards and negative inwards by convention,
CV is the control volume,
CS is the the control surface (enclosing the control volume),
$\vec{n}$ is a unit vector on each surface element $\mathrm{d}A$ pointing outwards,
$\vec{V}_{\text{rel}}$ is the local velocity of fluid relative to the control surface,
and $V_{\perp} \equiv \vec{V}_{\text{rel}} \cdot \vec{n}$ is the local cross-surface speed.

By inserting equations 3/2 and 3/3 into equation 3/1, we obtain:

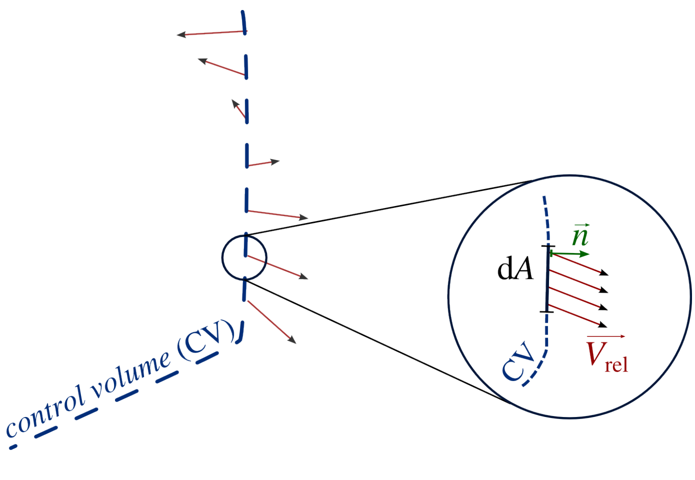


Figure 3.2 – Part of the system may be flowing through an arbitrary piece of the control surface with area $\mathrm{d}A$. The $\vec{n}$ vector defines the orientation of $\mathrm{d}A$ surface, and by convention is always pointed outwards.

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$$\boxed{\frac{\mathrm{d}B_{\text{sys}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{CV}} \rho b \,\mathrm{d}\mathcal{V} + \iint_{\text{CS}} \rho b \,(\vec{V}_{\text{rel}} \cdot \vec{n})\,\mathrm{d}A \qquad (3/4)}$$

Equation 3/4 is named the *Reynolds' transport theorem*; it stands now as a general, abstract accounting tool, but as we soon replace B by meaningful variables, it will prove extremely useful, allowing us to quantify the *net* effect of the flow of a system through a volume for which border properties are known.

In the following sections we are going to use this equation to assert four key physical principles (§0.7) in order to analyze the flow of fluids:

- mass conservation;
- change of linear momentum;
- change of angular momentum;
- energy conservation.

3.3 Mass conservation

Video: with sufficient skills (and lots of practice!), it is possible for a musician to produce an uninterrupted stream of air into an instrument while still continuing to breathe, a technique called *circular breathing*. Can you identify the different terms of eq. 3/5 as they apply to the clarinetist's mouth?

In this section, we focus on simply asserting that mass is conserved (eq. 0/22 p.19). Our study of the fluid's properties at the borders of the control volume is made by replacing variable B by mass m. Thus $\frac{\mathrm{d}B}{\mathrm{d}t}$ becomes $\frac{\mathrm{d}m_{\text{sys}}}{\mathrm{d}t}$, which by definition is zero.

In a similar fashion, $b \equiv B/m = m/m = 1$ and consequently the Reynolds transport theorem (3/4) becomes:

$$\frac{\mathrm{d}m_{\text{sys}}}{\mathrm{d}t} = 0 = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{CV}} \rho \,\mathrm{d}\mathcal{V} + \iint_{\text{CS}} \rho \,(\vec{V}_{\text{rel}} \cdot \vec{n})\,\mathrm{d}A \qquad (3/5)$$

$$\begin{matrix}\text{the time change} \\ \text{of the system's mass}\end{matrix} = 0 = \begin{matrix}\text{the rate of change} \\ \text{of mass inside} \\ \text{the control volume}\end{matrix} + \begin{matrix}\text{the net mass flow} \\ \text{at the borders} \\ \text{of the control volume}\end{matrix}$$

This equation 3/5 is often called *continuity equation*. It allows us to compare the incoming and outgoing mass flows through the borders of the control volume.

When the control volume has well-defined inlets and outlets through which the term $\rho(\vec{V}_{\text{rel}} \cdot \vec{n})$ can be considered uniform (as for example in fig. 3.3), this equation reduces to:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{CV}} \rho \,\mathrm{d}\mathcal{V} + \sum_{\text{out}} \{\rho V_\perp A\} + \sum_{\text{in}} \{\rho V_\perp A\} \qquad (3/6)$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{CV}} \rho \,\mathrm{d}\mathcal{V} + \sum_{\text{out}} \{\rho |V_\perp| A\} - \sum_{\text{in}} \{\rho |V_\perp| A\}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{CV}} \rho \,\mathrm{d}\mathcal{V} + \sum_{\text{out}} \{\dot{m}\} - \sum_{\text{in}} \{\dot{m}\} \qquad (3/7)$$

In equation 3/6, the term $\rho V_\perp A$ at each inlet or outlet corresponds to the local mass flow $\pm\dot{m}$ (positive inwards, negative outwards) through the boundary.

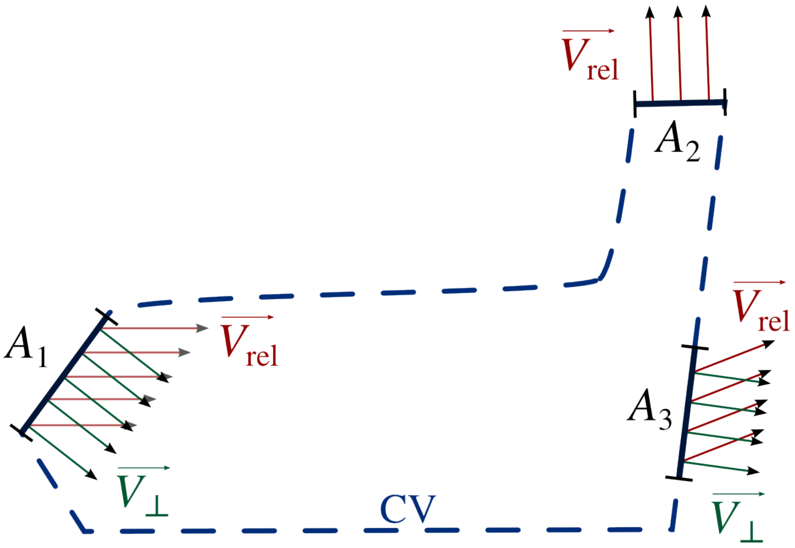

Figure 3.3 – A control volume for which the system's properties are uniform at each inlet and outlet. Here eq. 3/5 translates as $0 = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho \, \mathrm{d}\mathcal{V} + \rho_3 |V_{\perp 3}| A_3 + \rho_2 |V_{\perp 2}| A_2 - \rho_1 |V_{\perp 1}| A_1$.

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With equation 3/7 we can see that when the flow is steady (§0.8), the last two terms amount to zero, and the integral $\iiint_{\text{CV}} \rho \, \mathrm{d}\mathcal{V}$ (the total amount of mass in the control volume) does not change with time.

3.4 Change of linear momentum

In this section, we apply Newton's second law: we assert that the variation of system's linear momentum is equal to the net force being applied to it (eq. 0/23 p.19). Our study of the fluid's properties at the control surface is carried out by replacing variable B by the quantity $m\vec{V}$: momentum. Thus, $\frac{\mathrm{d}B_{\text{sys}}}{\mathrm{d}t}$ becomes $\frac{\mathrm{d}(m\vec{V}_{\text{sys}})}{\mathrm{d}t}$, which is equal to $\vec{F}_{\text{net}}$, the vector sum of forces applied on the system as it transits the control volume.
In a similar fashion, $b \equiv B/m = \vec{V}$ and equation 3/4, the Reynolds transport theorem, becomes:

$$\frac{\mathrm{d}(m\vec{V}_{sys})}{\mathrm{d}t} = \vec{F}_{\text{net}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho \vec{V} \, \mathrm{d}\mathcal{V} + \iint_{\text{CS}} \rho \vec{V} \, (\vec{V}_{\text{rel}} \cdot \vec{n}) \, \mathrm{d}A \tag{3/8}$$

the vector sum of forces on the system = the rate of change of linear momentum within the control volume + the net flow of linear momentum through the boundaries of the control volume

When the control volume has well-defined inlets and outlets through which the term $\rho\vec{V}(\vec{V}_{\text{rel}} \cdot \vec{n})$ can be considered uniform (fig. 3.3), this equation reduces to:

$$\vec{F}_{\text{net}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho \vec{V} \, \mathrm{d}\mathcal{V} + \sum_{\text{out}} \left\{ (\rho |V_\perp| A) \vec{V} \right\} - \sum_{\text{in}} \left\{ (\rho |V_\perp| A) \vec{V} \right\} \tag{3/9}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho \vec{V} \, \mathrm{d}\mathcal{V} + \sum_{\text{out}} \left\{ \dot{m}\vec{V} \right\} - \sum_{\text{in}} \left\{ \dot{m}\vec{V} \right\} \tag{3/10}$$

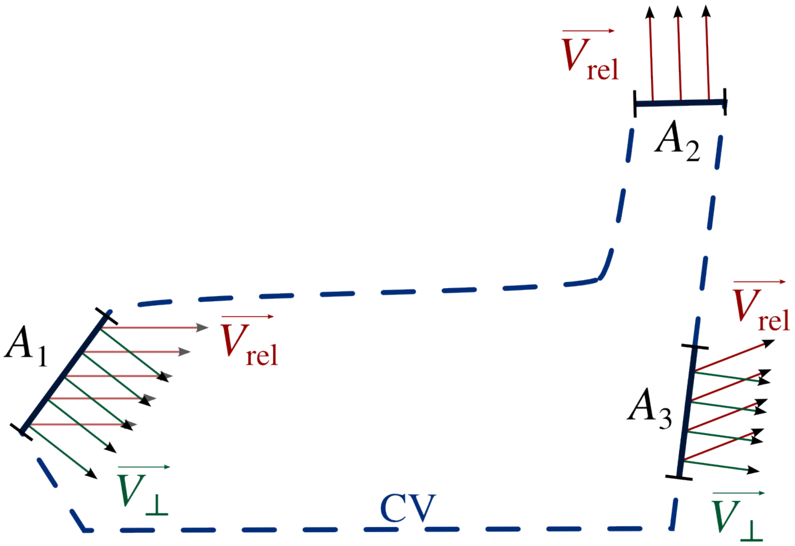

Figure 3.4 – The same control volume as in fig. 3.3. Here, since the system's properties are uniform at each inlet and outlet, eq. 3/8 translates as $\vec{F}_{\text{net}} = \frac{d}{dt} \iiint_{CV} \rho \vec{V} \, d\mathcal{V} + \rho_3 |V_{\perp 3}| A_3 \vec{V}_3 + \rho_2 |V_{\perp 2}| A_2 \vec{V}_2 - \rho_1 |V_{\perp 1}| A_1 \vec{V}_1$.

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Let us observe the four terms of equation 3/10 for a moment, for they are full of subtleties.

First, we notice that even if the flow is steady (an therefore that $\sum_{\text{net}} (\dot{m}) = 0 \text{ kg s}^{-1}$ since $\frac{d}{dt} = 0$), the last two terms do not necessarily cancel each other (i.e. it is possible that $\sum_{\text{net}} \left(\dot{m}\vec{V}\right) \neq \vec{0}$).

The reverse also applies: it is quite possible that $\vec{F}_{\text{net}} \neq \vec{0}$ even if the net momentum flow through the boundaries is null (that is, even if $\sum_{\text{net}} \left(\dot{m}\vec{V}\right) = \vec{0}$). This would be the case, if $\iiint_{CV} \rho \, d\mathcal{V}$ (the total amount of momentum within the borders of the control volume) varies with time. Walking forwards and backwards within a rowboat would cause such an effect.

Video: as a person walks, the deflection of the air passing around their body can be used to sustain the flight of a paper airplane (a *walkalong glider*). Can you figure out the momentum flow entering and leaving a control volume surrounding the glider?

by Y:sciencetoymaker (STYL)
https://youtu.be/S6JKwzK37_8

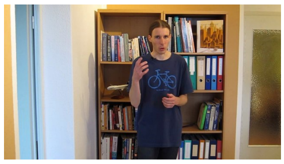
From equation 3/10 therefore, we read that two distinct phenomena can result in a net force on the system:

- A difference between the values of $\dot{m}\vec{V}$ of the fluid at the entrance and exit of the control volume (caused, for example, by a deviation of the flow or by a mass flow imbalance);
- A change in time of the momentum $m\vec{V}$ within the control volume (for example, with the acceleration or the variation of the mass of the control volume).

These two factors may cancel each other, so that the system may well be able to travel through the control volume without any net force being applied to it.

3.5 Change of angular momentum

In this third spin on the Reynolds transport theorem, we assert that the change of the angular momentum of a system about a point X is equal to the net moment applied on the system about this point (eq. 0/24 p.19). Our study of the fluid's properties at the borders of the control volume is made by replacing the variable B by the angular momentum $\vec{r}_{\mathrm{X}m} \wedge m\vec{V}$. Thus, $\frac{\mathrm{d}B_{\mathrm{sys}}}{\mathrm{d}t}$ becomes $\frac{\mathrm{d}(\vec{r}_{\mathrm{X}m} \wedge m\vec{V}_{\mathrm{sys}})}{\mathrm{d}t}$, which is equal to $\vec{M}_{\mathrm{net}}$, the vector sum of moments applied on the system about point X as it transits through the control volume.



Video: pre-lecture briefing for this chapter, part 2/2

by o.c. (CC-BY)
https://youtu.be/nmEe7Dq01AU

In a similar fashion, $b \equiv B/m = \vec{r} \wedge \vec{V}$ and equation 3/4, the Reynolds transport theorem, becomes:

$$\frac{\mathrm{d}(\vec{r}_{\mathrm{X}m} \wedge m\vec{V})_{\mathrm{sys}}}{\mathrm{d}t} = \vec{M}_{\mathrm{net,X}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\mathrm{CV}} \vec{r}_{\mathrm{X}m} \wedge \rho \vec{V} \, \mathrm{d}\mathcal{V} + \iint_{\mathrm{CS}} \vec{r}_{\mathrm{X}m} \wedge \rho \, (\vec{V}_{\mathrm{rel}} \cdot \vec{n}) \vec{V} \, \mathrm{d}A \tag{3/11}$$

the sum of moments applied to the system = the rate of change of the angular momentum in the control volume + the net flow of angular momentum through the control volume's boundaries

in which $\vec{r}_{\mathrm{X}m}$ is a vector giving the position of any mass m relative to point X.

When the control volume has well-defined inlets and outlets through which the term $\vec{r}_{\mathrm{X}m} \wedge \rho \, (\vec{V}_{\mathrm{rel}} \cdot \vec{n}) \vec{V}$ can be considered uniform (fig. 3.5), this equation reduces to:

Video: rocket landing gone wrong. Can you compute the moment exerted by the top thruster around the base of the rocket as it (unsuccessfully) attempts to compensate for the collapsed landing leg?

by Y:SciNews (STYL)
https://youtu.be/4cvGGxTsQx0

$$\vec{M}_{\mathrm{net,X}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\mathrm{CV}} \vec{r}_{\mathrm{X}m} \wedge \rho \vec{V} \, \mathrm{d}\mathcal{V} + \sum_{\mathrm{out}} \left\{ \vec{r}_{\mathrm{X}m} \wedge \dot{m} \vec{V} \right\} - \sum_{\mathrm{in}} \left\{ \vec{r}_{\mathrm{X}m} \wedge \dot{m} \vec{V} \right\} \tag{3/12}$$

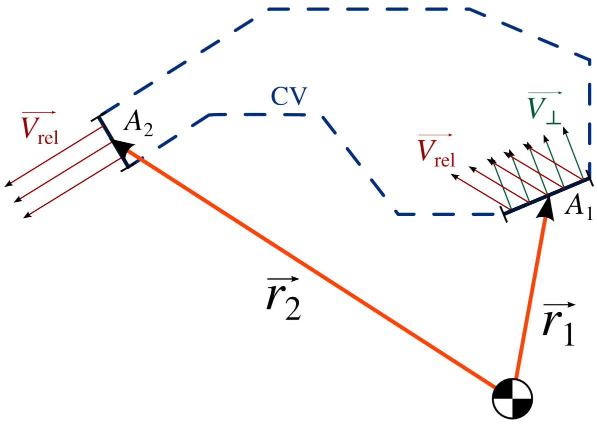


Figure 3.5 – A control volume for which the properties of the system are uniform at each inlet or outlet. Here the moment about point X is $\vec{M}_{\mathrm{net,X}} \approx \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\mathrm{CV}} \vec{r}_{\mathrm{X}m} \wedge \rho \vec{V} \, \mathrm{d}\mathcal{V} + \vec{r}_2 \wedge |\dot{m}_2| \vec{V}_2 - \vec{r}_1 \wedge |\dot{m}_1| \vec{V}_1$.

Figure CC-0 o.c.

Equation 3/12 allows us to quantify, with relative ease, the moment exerted on a system based on inlet and outlet velocities of a control volume.

3.6 Energy conservation

We conclude our frantic exploration of control volume analysis with the first principle of thermodynamics. We now simply assert that the change in the energy of a system can only be due to well-identified transfers (eq. 0/25 p.19). Our study of the fluid's properties at the borders of the control volume is made by replacing variable B by an amount of energy E_{sys}. Now, $\frac{\mathrm{d}E_{\text{sys}}}{\mathrm{d}t}$ can be attributed to three contributors:

$$\frac{\mathrm{d}E_{\text{sys}}}{\mathrm{d}t} = \dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} + \dot{W}_{\text{pressure, net in}} \tag{3/13}$$

where $\dot{Q}_{\text{net in}}$ is the net power transfered as heat;
$\dot{W}_{\text{shaft, net in}}$ is the net power added as work with a shaft;
and $\dot{W}_{\text{pressure, net in}}$ is the net power required to enter and leave the control volume.

It follows that $b \equiv B/m = E/m \equiv e$; and e is broken down into

$$e = i + e_k + e_p \tag{3/14}$$

where i is the specific internal energy ($\mathrm{J\,kg^{-1}}$);
e_k the specific kinetic energy ($\mathrm{J\,kg^{-1}}$);
and e_p the specific potential energy ($\mathrm{J\,kg^{-1}}$).

Now, the Reynolds transport theorem (equation 3/4) becomes:

$$\frac{\mathrm{d}E_{\text{sys}}}{\mathrm{d}t} = \dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} + \dot{W}_{\text{pressure, net in}} = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{CV}} \rho\, e\, \mathrm{d}\mathcal{V} + \iint_{\text{CS}} \rho\, e\, (\vec{V}_{\text{rel}} \cdot \vec{n})\, \mathrm{d}A \tag{3/15}$$

When the control volume has well-defined inlets and outlets through which the term $\rho\, e\, (\vec{V}_{\text{rel}} \cdot \vec{n})$ can be considered uniform, this equation reduces to:

$$\begin{aligned}
\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} + \dot{W}_{\text{pressure, net in}} &= \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{CV}} \rho\, e\, \mathrm{d}\mathcal{V} + \sum_{\text{out}} \{\dot{m}\, e\} - \sum_{\text{in}} \{\dot{m}\, e\} \qquad (3/16)\\
\dot{Q}_{\text{netin}} + \dot{W}_{\text{shaft, net in}} &= \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{CV}} \rho\, e\, \mathrm{d}\mathcal{V} + \sum_{\text{out}} \left\{\dot{m}(i + e_k + e_p)\right\} \\
&\quad - \sum_{\text{in}} \left\{\dot{m}(i + e_k + e_p)\right\} - \dot{W}_{\text{pressure, net in}} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{CV}} \rho\, e\, \mathrm{d}\mathcal{V} + \sum_{\text{out}} \left\{\dot{m}\left(i + \frac{1}{2}V^2 + gz\right)\right\} \\
&\quad - \sum_{\text{in}} \left\{\dot{m}\left(i + \frac{1}{2}V^2 + gz\right)\right\} + \sum_{\text{out}} \left\{\dot{m}\frac{p}{\rho}\right\} - \sum_{\text{in}} \left\{\dot{m}\frac{p}{\rho}\right\}
\end{aligned}$$

Making use of the concept of *enthalpy* defined as $h \equiv i + p/\rho$, we obtain:

$$\dot{Q}_{\text{net in}} + \dot{W}_{\text{shaft, net in}} = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{CV}} \rho\, e\, \mathrm{d}\mathcal{V} + \sum_{\text{out}} \left\{\dot{m}\left(h + \frac{1}{2}V^2 + gz\right)\right\} - \sum_{\text{in}} \left\{\dot{m}\left(h + \frac{1}{2}V^2 + gz\right)\right\} \tag{3/17}$$

the net power received by the system = the rate of change of energy inside the control volume + the net energy flow rate exiting the control volume boundaries

This equation 3/17 is particularly attractive, but it necessitates the input of a large amount of experimental data to provide useful results. It is indeed very difficult to predict how the terms i and p/ρ will change for a given flow process. For example, a pump with given power $\dot{Q}_{\text{net in}}$ and $\dot{W}_{\text{shaft, net in}}$ will generate large increases of terms p, V and z if it is efficient, or a large increase of terms i and $1/\rho$ if it is inefficient. This equation 3/17, sadly, does not allow us to quantify the net effect of shear and the extent of irreversibilities in a fluid flow.

3.7 The Bernoulli equation

The Bernoulli equation has very little practical use for us; nevertheless it is so widely used that we have to dedicate a brief section to examining it. We will start from equation 3/17 and add five constraints:

1. Steady flow.
   Thus $\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{CV}} \rho\, e \,\mathrm{d}\mathcal{V} = 0$.
   In addition, $\dot{m}$ has the same value at inlet and outlet;

2. Incompressible flow.
   Thus, ρ stays constant;

3. No heat or work transfer.
   Thus, both $\dot{Q}_{\text{net in}}$ and $\dot{W}_{\text{shaft, net in}}$ are zero;

4. No friction.
   Thus, the fluid energy i cannot increase due to an input from the control volume;

5. One-dimensional flow.
   Thus, our control volume has only one known entry and one known exit, all fluid particles move together with the same transit time, and the overall trajectory is already known.

With these five restrictions, equation 3/17 simply becomes:

$$\begin{aligned}
0 &= \sum_{\text{out}} \left\{ \dot{m}(i + \frac{p}{\rho} + e_k + e_p) \right\} - \sum_{\text{in}} \left\{ \dot{m}(i + \frac{p}{\rho} + e_k + e_p) \right\} \\
&= \dot{m} \left[(i + \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2) - (i + \frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1) \right]
\end{aligned}$$

and we here obtain the *Bernoulli equation*:

$$\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + gz_2 \tag{3/18}$$

This equation describes the properties of a fluid particle in a steady, incompressible, friction-less flow with no energy transfer.

The Bernoulli equation can also be obtained starting from the linear momentum equation (eq. 3/8 p.65):

$$\vec{F}_{\text{net}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho \vec{V} \,\mathrm{d}\mathcal{V} + \iint_{\text{CS}} \rho \vec{V} \,(\vec{V}_{\text{rel}} \cdot \vec{n}) \,\mathrm{d}A$$

When considering a fixed, infinitely short control volume along a known streamline s of the flow, this equation becomes:

$$\mathrm{d}\vec{F}_{\text{pressure}} + \mathrm{d}\vec{F}_{\text{shear}} + \mathrm{d}\vec{F}_{\text{gravity}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho \vec{V} \, \mathrm{d}\mathcal{V} + \rho V A \, \mathrm{d}\vec{V}$$

along a streamline, where the velocity $\vec{V}$ is aligned (by definition) with the streamline.

Now, adding the restrictions of steady flow ($\mathrm{d}/\mathrm{d}t = 0$) and no friction ($\mathrm{d}\vec{F}_{\text{shear}} = \vec{0}$), we already obtain:

$$\mathrm{d}\vec{F}_{\text{pressure}} + \mathrm{d}\vec{F}_{\text{gravity}} = \rho V A \, \mathrm{d}\vec{V}$$

The projection of the net force due to gravity $\mathrm{d}\vec{F}_{\text{gravity}}$ on the streamline segment $\mathrm{d}s$ has norm $\mathrm{d}\vec{F}_{\text{gravity}} \cdot \mathrm{d}\vec{s} = -g\rho A \, \mathrm{d}z$, while the net force due to pressure is aligned with the streamline and has norm $\mathrm{d}F_{\text{pressure},s} = -A \, \mathrm{d}p$. Along this streamline, we thus have the following scalar equation, which we integrate from points 1 to 2:

$$-A \, \mathrm{d}p - \rho g A \, \mathrm{d}z = \rho V A \, \mathrm{d}V$$

$$-\frac{1}{\rho} \, \mathrm{d}p - g \, \mathrm{d}z = V \, \mathrm{d}V$$

$$-\int_1^2 \frac{1}{\rho} \, \mathrm{d}p - \int_1^2 g \, \mathrm{d}z = \int_1^2 V \, \mathrm{d}V$$

The last obstacle is removed when we consider flows without heat or work transfer, where, therefore, the density ρ is constant. In this way, we arrive to equation. 3/18 again:

$$\frac{p_1}{\rho} + \frac{1}{2} V_1^2 + g z_1 = \frac{p_2}{\rho} + \frac{1}{2} V_2^2 + g z_2$$

Let us insist on the incredibly frustrating restrictions brought by the five conditions above:

1. Steady flow.
   This constrains us to continuous flows with no transition effects, which is a reasonable limit;

2. Incompressible flow.
   We cannot use this equation to describe flow in compressors, turbines, diffusers, nozzles, nor in flows where $M > 0{,}6$.

3. No heat or work transfer.
   We cannot use this equation in a machine (e.g. in pumps, turbines, combustion chambers, coolers).

4. No friction.
   This is a tragic restriction! We cannot use this equation to describe a turbulent or viscous flow, e.g. near a wall or in a wake.

5. One-dimensional flow.
   This equation is only valid if we know precisely the trajectory of the fluid whose properties are being calculated.

Among these, the last is the most severe (and the most often forgotten): **the Bernoulli equation does not allow us to predict the trajectory of fluid particles**. Just like all of the other equations in this chapter, it requires a control volume with a known inlet and a known outlet.

3.8 Limits of integral analysis

Integral analysis is an incredibly useful tool in fluid dynamics: in any given problem, it allows us to rapidly describe and calculate the main fluid phenomena at hand. The net force exerted on the fluid as it is deflected downwards by a helicopter, for example, can be calculated using just a loosely-drawn control volume and a single vector equation.

As we progress through exercise sheet 3, however, the limits of this method slowly become apparent. They are twofold.

- First, we are confined to calculating the *net* effect of fluid flow. The net force, for example, encompasses the integral effect of all forces —due to pressure, shear, and gravity— applied on the fluid as it transits through the control volume. Integral analysis gives us absolutely no way of distinguishing between those sub-components. In order to do that (for example, to calculate which part of a pump's mechanical power is lost to internal viscous effects), we would need to look within the control volume.

- Second, all four of our equations in this chapter only work in one direction. The value $\mathrm{d}B_{\text{sys}}/\,\mathrm{d}t$ of any finite integral cannot be used to find which function $\rho b V_\perp \,\mathrm{d}A$ was integrated over the control surface to obtain it. For example, there are an *infinite* number of velocity profiles which will result in a net force of $-12\,\mathrm{N}$. Knowing the net value of an integral, we cannot deduce the conditions which lead to it.
  In practice, this is a major limitation on the use of integral analysis, because it confines us to working with large swaths of experimental data gathered at the borders of our control volumes. From the wake below the helicopter, we deduce the net force; but the net force tells us nothing about the shape of the wake.

Clearly, in order to overcome these limitations, we are going to need to open up the control volume, and look at the details of the flow within — perhaps by dividing it into a myriad of sub-control volumes. This is what we set ourselves to in chapter 4, with a thundering and formidable methodology we shall call *derivative analysis*.