# Fluid Dynamics
# Chapter 3 – Analysis of existing flows with three dimensions

last edited May 22, 2020
by Olivier Cleynen — https://fluidmech.ninja/



These notes are based on textbooks by White [22], Çengel & al.[25], Munson & al.[29], and de Nevers [17].

## 3.1 Motivation

In this chapter, we use the same tools that we developed in chapter 2, but we improve them so we can apply them to more complex cases. Specifically, we would like to answer the following questions:

1. What are the mass flows and forces involved when a flow has non-uniform velocity?
2. What are the forces and moments involved when a flow changes direction?

## 3.2 The Reynolds transport theorem

### 3.2.1 Control volume

Let us begin, this time, by building a control volume in *any arbitrary flow*: we are no longer limited to one-inlet, one-oulet steady-flow situations. Instead, we will write equations that work inside any generic *velocity field* $\vec{V} = (u, v, w)$ which is a function of space and time: $\vec{V} = f(x, y, z, t)$.

Within this flow, we draw an arbitrary volume named *control volume* (CV) which is free to move and change shape (fig. 3.1). We are going to measure the properties of the fluid at the borders of this volume, which we call the *control surface* (CS), in order to compute the net effect of the flow through the volume.

At a given time, the control volume contains a certain amount of mass which we call the *system* (sys). Thus the system is a fixed amount of mass transiting

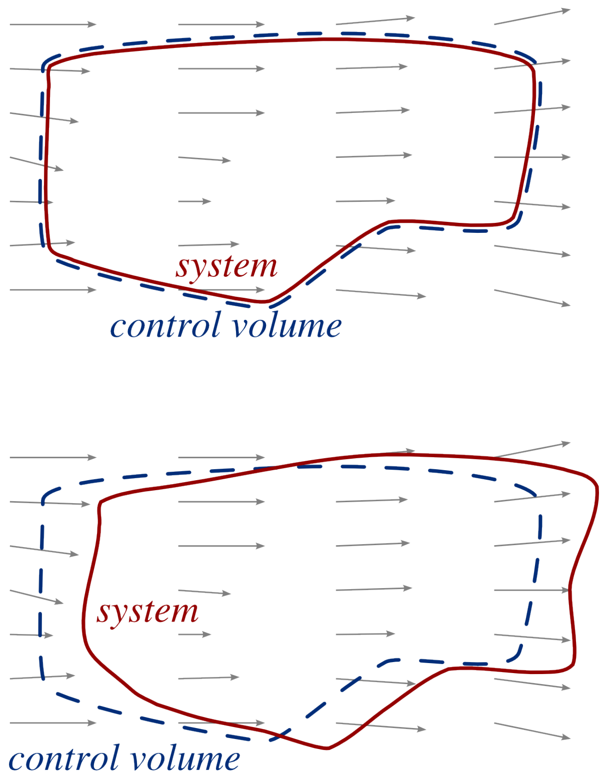



Figure 3.1: A control volume within an arbitrary flow (compare with figure 2.1 p. 34). The *system* is the amount of mass included within the control volume at a given time. At a later time (bottom), it may have left the control volume, and its shape and properties may have changed. The control volume may also change shape with time, although this is not represented here.

*Figure* CC-0 *Olivier Cleynen*

through the control volume at the exact time when we write the equation, and its properties (volume, pressure, velocity etc.) may change in the process.

All along the chapter, we are focusing on the question: based on measured fluid properties at some chosen area in space and time (the properties at the control surface), how can we quantify what is happening to the system (the mass inside the control volume)?

### 3.2.2 Rate of change of an additive property

In order to proceed with our calculations, we need to make our accounting methodology even more robust. We start with a "dummy" fluid property $B$, which we will later replace with physical variables of interest.
Let us therefore consider an arbitrary additive property $B$ of the fluid. By the term *additive* property, we mean that the total *amount of property* is divided if the fluid is divided. For instance, this works for properties such as mass, volume, energy, entropy, but not pressure or temperature.
The *specific* (i.e. per unit mass) value of $B$ is designated $b \equiv B/m$.

We now want to compute the variation of a system's property $B$ based on measurements made at the borders of the control volume. We will achieve this with an equation containing three terms:

- The time variation of the quantity $B$ within the system is measured with the term $\mathrm{d}B_{\text{sys}}/\,\mathrm{d}t$.
  We will use this term to represent, for example, the rate of change of the fluid's energy or momentum as it travels through a jet engine.

- Within the control volume, the enclosed quantity $B_{\text{CV}}$ can vary by accumulation (for example, mass may be increasing in an air tank fed with compressed air): we measure this with the term $\mathrm{d}B_{\text{CV}}/\,\mathrm{d}t$.

- Finally, a mass flux may be flowing through the boundaries of the control volume, carrying with it some amount of $B$ every second: we write that net flow out of the system as $\dot{B}_{\text{net}} \equiv \dot{B}_{\text{out}} - \dot{B}_{\text{in}}$.

We can now link these three terms with the simple equation:

$$\frac{\mathrm{d}B_{\text{sys}}}{\mathrm{d}t} = \frac{\mathrm{d}B_{\text{CV}}}{\mathrm{d}t} + \dot{B}_{\text{net}} \tag{3/1}$$

$$\begin{array}{ccccc} \text{the rate of change} & & \text{the rate of change} & & \text{the net flow of } B \\ \text{of the fluid's } B & = & \text{of } B \text{ inside} & + & \text{at the borders} \\ \text{as it transits} & & \text{considered volume} & & \text{of the considered volume} \end{array}$$

Since $B$ may not be uniformly distributed within the control volume, we like to express the term $B_{\text{CV}}$ as the integral of the volume density $B/\mathcal{V}$ with respect to volume $\mathcal{V}$:

$$B_{\text{CV}} = \iiint_{\text{CV}} \frac{B}{\mathcal{V}} \,\mathrm{d}\mathcal{V} = \iiint_{\text{CV}} \rho b \,\mathrm{d}\mathcal{V} \tag{3/2}$$

$$\frac{\mathrm{d}B_{\text{CV}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho b \,\mathrm{d}\mathcal{V} \tag{3/3}$$

where CV is the control volume,
and $\mathcal{V}$ is volume ($\text{m}^3$).

The second term of eq. 3/1, $\dot{B}_{\text{net}}$, can be evaluated by quantifying, for each area element $\mathrm{d}A$ of the control volume's surface, the surface flow rate $\rho b V_\perp$ of property $B$ that flows through it, as shown in fig. 3.2. The integral over the entire control volume surface CS of this term is:

$$\dot{B}_{\text{net}} = \iint_{\text{CS}} \rho b V_\perp \,\mathrm{d}A = \iint_{\text{CS}} \rho b \left(\vec{V}_{\text{rel}} \cdot \vec{n}\right) \mathrm{d}A \tag{3/4}$$

where CS is the control surface (enclosing the control volume CV),
$\vec{n}$ is a unit vector on each surface element $\mathrm{d}A$ pointing outwards,
$\vec{V}_{\text{rel}}$ is the local velocity of fluid relative to the control surface,
and $V_\perp \equiv \vec{V}_{\text{rel}} \cdot \vec{n}$ is the local cross-surface speed (positive outwards, neg. inwards)

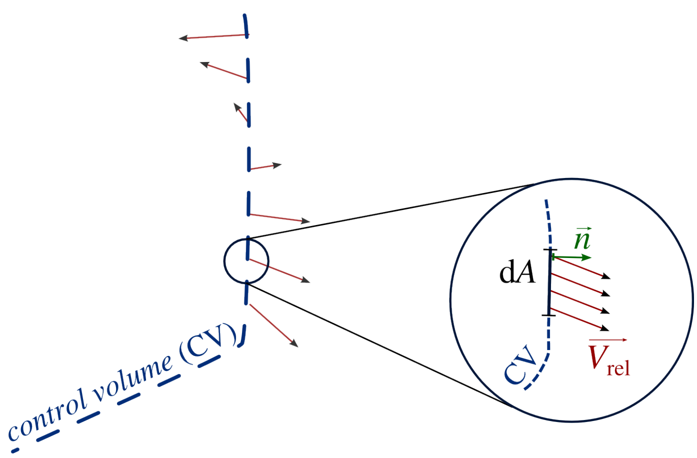



Figure 3.2: Part of the system may be flowing through an arbitrary piece of the control surface with area $\mathrm{d}A$. The $\vec{n}$ vector defines the orientation of $\mathrm{d}A$ surface, and by convention is always pointed outwards. The amount of $B$ flowing through this small area per unit time is $\mathrm{d}A$ is $\rho V_\perp \,\mathrm{d}Ab$

By inserting equations 3/3 and 3/4 into equation 3/1, we obtain:

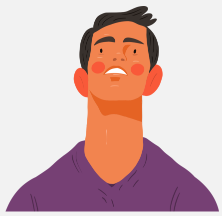
$$\frac{\mathrm{d}B_{\text{sys}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho b \,\mathrm{d}\mathcal{V} + \iint_{\text{CS}} \rho b \,(\vec{V}_{\text{rel}} \cdot \vec{n})\,\mathrm{d}A \tag{3/5}$$

Equation 3/5 is named the *Reynolds transport theorem*[W]; it stands now as a general, abstract accounting tool, but as we soon replace $B$ by meaningful variables, it will prove extremely useful, allowing us to quantify the *net* effect of the flow of a system through a volume for which border properties are known.

In the following sections we are going to use this equation to write out four key balance equations (see §1.7 p. 20):

- balance of mass;
- balance of linear momentum;
- balance of angular momentum;
- balance of energy.

*Advice from an expert*



Take some time to observe the (sometimes curious!) terms of equation 3/5, and learn the associated vocabulary. Just like an accountant in a company would like to have a very clear method for counting how money is spent and earned, fluid dynamicists need very good tools to describe what's coming in and out of their flows. When you'll be hurling three-dimensional vector operations at swirling flows down this chapter, you'll be glad you learned about control surfaces and sign conventions earlier on.

## 3.3 Balance of mass

What is the balance of mass for fluid flowing through any arbitrary volume? We answer this question by writing out a mass balance equation in the template provided by the Reynolds transport theorem (eq. 3/5).

We now state that the placeholder variable $B$ is mass $m$. It follows that $\mathrm{d}B/\mathrm{d}t$ becomes $\mathrm{d}m_{\text{sys}}/\mathrm{d}t$, which by definition is zero (see eq. 1/24 p. 20). Also, $b \equiv B/m = m/m = 1$ and now the Reynolds transport theorem becomes:

$$\begin{array}{ccccc} \dfrac{\mathrm{d}m_{\text{sys}}}{\mathrm{d}t} & = 0 = & \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\iiint_{\text{CV}} \rho \,\mathrm{d}\mathcal{V} & + & \displaystyle\iint_{\text{CS}} \rho \,(\vec{V}_{\text{rel}} \cdot \vec{n})\,\mathrm{d}A \\ \begin{array}{c}\text{the rate of change}\\ \text{of the fluid's mass}\\ \text{as it transits}\end{array} & = 0 = & \begin{array}{c}\text{the rate of change}\\ \text{of mass inside}\\ \text{considered volume}\end{array} & + & \begin{array}{c}\text{the net mass flow}\\ \text{at the borders}\\ \text{of the considered volume}\end{array} \end{array} \tag{3/6}$$

This equation 3/6 is often called *continuity equation*. It allows us to compare the incoming and outgoing mass flows through the borders of the control volume.

Video: with sufficient skills (and lots of practice!), it is possible for a musician to produce an uninterrupted stream of air into an instrument while still continuing to breathe, a technique called circular breathing.[W] Can you identify the different terms of eq. 3/8 as they apply to the saxophonist's mouth?



Sometimes, the control volume has well-defined inlets and outlets through which the volume flow $\rho(\vec{V}_{\text{rel}} \cdot \vec{n})$ is uniform: this is illustrated in figure 3.3. In that case equation 3/6 reduces to forms that we have already identified in the previous chapter (see §2.3 p. 35):

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho \, \mathrm{d}\mathcal{V} + \sum_{\text{out}} \{\rho V_\perp A\} + \sum_{\text{in}} \{\rho V_\perp A\} \tag{3/7}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho \, \mathrm{d}\mathcal{V} + \sum_{\text{out}} \{\rho |V_\perp| A\} - \sum_{\text{in}} \{\rho |V_\perp| A\}$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} m_{\text{CV}} + \sum_{\text{out}} \{|\dot{m}|\} - \sum_{\text{in}} \{|\dot{m}|\} \tag{3/8}$$

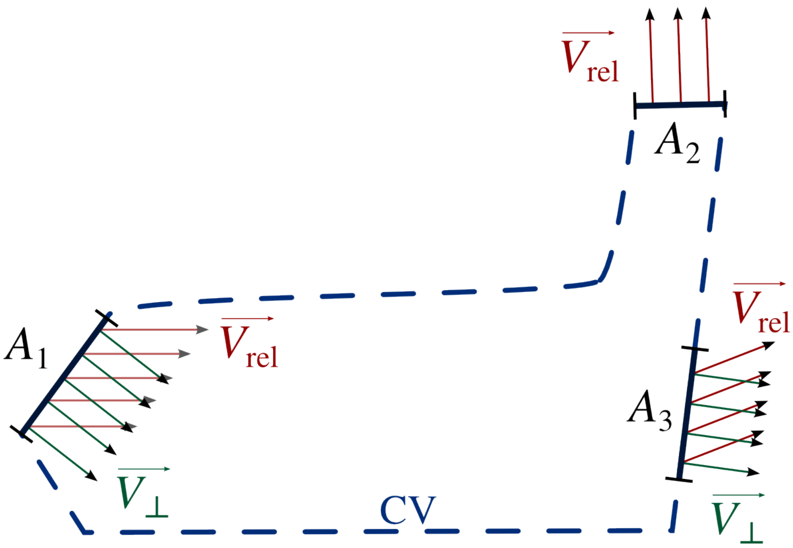



Figure 3.3: A control volume for which the system's properties are uniform at each inlet and outlet. Here eq. 3/6 translates as $0 = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho \, \mathrm{d}\mathcal{V} + \rho_3 |V_{\perp 3}| A_3 + \rho_2 |V_{\perp 2}| A_2 - \rho_1 |V_{\perp 1}| A_1$.

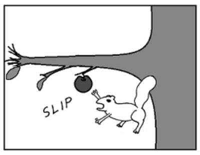
*Figure* CC-0 *Olivier Cleynen*

As before, in equation 3/7, the term $\rho V_\perp A$ at each inlet or outlet corresponds to the local mass flow $\pm \dot{m}$ (negative inwards, positive outwards) through the boundary.

*Advice from an expert*

If you are thinking that equation 3/6 is just a very fancy way to write the "good" mass balance equation we already wrote in chapter 2 (eq. 2/3 p. 35), you are not completely wrong. There are two useful improvements, however:
First, having $\iint \rho V_\perp \, \mathrm{d}A$ instead of just $\rho V_\perp A$ allows us to handle cases where the incoming/outgoing velocities are not uniform. Outside of fluid mechanics textbooks, very few flows have a nice smooth uniform outlet!
Second, there is an unsteady term $\mathrm{d}m_{\text{CV}}/\,\mathrm{d}t$. It is not used to solve problems in this course, but one day when you are confronted to a case where your inlet and outlet mass flows are not equal, it will save your day!

## 3.4 Balance of momentum




Abstruse Goose #338: Newton's laws of motion almost didn't happen

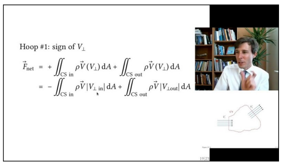
by an anonymous artist (CC-BY-NC)
https://abstrusegoose.com/338

What force is applied to the fluid for it to travel through the control volume? We answer this question by writing out a mass balance equation in the template provided by the Reynolds transport theorem (eq. 3/5).

We now state that the placeholder variable $B$ is momentum $m\vec{V}$. It follows that $\mathrm{d}B/\mathrm{d}t$ becomes $\mathrm{d}m\vec{V}_{\text{sys}})/\mathrm{d}t$, which by definition is the net force $\vec{F}_{\text{net}}$ applying to the system (see eq. 1/25 p. 20). Also, $b \equiv B/m = m\vec{V}/m = \vec{V}$ and now the Reynolds transport theorem becomes:

$$\frac{\mathrm{d}(m\vec{V}_{\text{sys}})}{\mathrm{d}t} = \vec{F}_{\text{net}} = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{CV}} \rho\vec{V}\,\mathrm{d}\mathcal{V} + \iint_{\text{CS}} \rho\vec{V}\,(\vec{V}_{\text{rel}}\cdot\vec{n})\,\mathrm{d}A \tag{3/9}$$

the vector sum of forces on the fluid = the rate of change of momentum within the considered volume + the net flow of momentum through the boundaries of the considered volume



Video: Making sense of the 3D linear momentum balance equation

by Olivier Cleynen (CC-BY)
https://youtu.be/iDCpqoJJSI4

Sometimes, the control volume has well-defined inlets and outlets through which the flow $\rho\vec{V}(\vec{V}_{\text{rel}}\cdot\vec{n})$ is uniform: this is illustrated again in figure 3.4. In that case equation 3/9 reduces to forms that we have already identified in the previous chapter (see §2.4 p. 37):

$$\vec{F}_{\text{net}} = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{CV}} \rho\vec{V}\,\mathrm{d}\mathcal{V} + \sum_{\text{out}}\left\{(\rho|V_\perp|A)\vec{V}\right\} - \sum_{\text{in}}\left\{(\rho|V_\perp|A)\vec{V}\right\} \tag{3/10}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(m\vec{V}\right)_{\text{CV}} + \sum_{\text{out}}\left\{|\dot{m}|\vec{V}\right\} - \sum_{\text{in}}\left\{|\dot{m}|\vec{V}\right\} \tag{3/11}$$

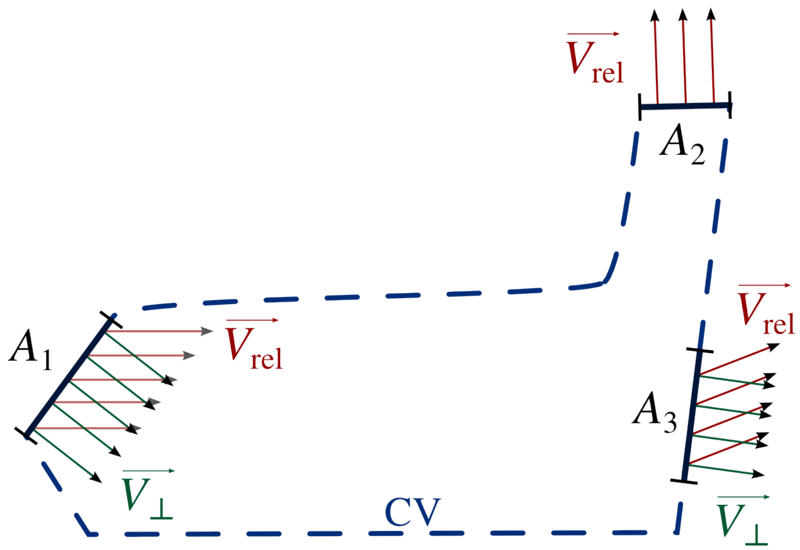



Figure 3.4: The same control volume as in fig. 3.3. Here, since the system's properties are uniform at each inlet and outlet, eq. 3/9 translates as $\vec{F}_{\text{net}} = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\text{CV}} \rho\vec{V}\,\mathrm{d}\mathcal{V} + \rho_3|V_{\perp 3}|A_3\vec{V}_3 + \rho_2|V_{\perp 2}|A_2\vec{V}_2 - \rho_1|V_{\perp 1}|A_1\vec{V}_1$.

*Figure* CC-0 *Olivier Cleynen*

To make clear a few things, let us focus on the simple case where a considered volume has only one inlet (point 1) and one outlet (point 2). From equation 3/9, the net force $\vec{F}_{\text{net}}$ applying on the fluid is:

$$\vec{F}_{\text{net}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho \vec{V} \,\mathrm{d}\mathcal{V} + \iint \rho_2 |V_{\perp 2}| \vec{V}_2 \,\mathrm{d}A_2 - \iint \rho_1 |V_{\perp 1}| \vec{V}_1 \,\mathrm{d}A_1 \tag{3/12}$$


Video: as a person walks, the deflection of the air passing around their body can be used to sustain the flight of a paper airplane (a *walkalong glider*[W]). Can you figure out the momentum flow entering and leaving a control volume surrounding the glider, and the resulting net force?
by Y:sciencetoymaker (STYL)
https://youtu.be/S6JKwzK37_8

In this equation 3/12, what could cause $\vec{F}_{\text{net}}$ to be non-zero?

- The first term, which we could informally write as $\mathrm{d}(m\vec{V})_{\text{CV}}/\mathrm{d}t$, could be non-zero. This happens when the momentum inside the control volume changes. This may occur if the distribution of velocities $\vec{V}$ within the control volume is changing, such as when the fluid in a tank sloshes back and forth against the walls.
- The sum of the last two terms, which we could informally write as $|\dot{m}|_2 \vec{V}_2 - |\dot{m}|_1 \vec{V}_1$, could also be non-zero. This happens when the flux of momentum entering the control volume is different from the one leaving it:
  - It may be because the mass flow $\dot{m}$ is different at inlet and outlet, even if the two velocity distributions are the same;
  - It may be because the velocities have different length, and the flow is speeding up or slowing down;
  - It may be because the velocities are aligned differently, and the flow is changing directions;
  - It may be because the velocities are non-uniform and distributed differently, even if their average (and thus the mass flows) are the same.

As you can see, a lot of different things may be happening at once! We will study (separately) the most relevant of those effects in the problem section of this chapter.

## 3.5 Balance of angular momentum

What moment ("twisting effort") is applied to the fluid for it to travel through the control volume? We answer this question by writing an angular momentum balance (see eq. 1/26 p. 20) in the template provided by the Reynolds transport theorem (eq. 3/5).

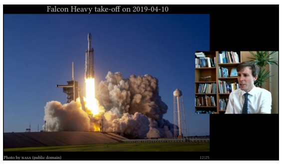

Video: rocket landing gone wrong. Can you compute the moment exerted by the top thruster around the base of the rocket as it (unsuccessfully) attempts to compensate for the collapsed landing leg?
by Y:SciNews (STYL)
https://youtu.be/4cvGGxTsQx0

We position ourselves at a point $X$, about which we measure all moments. All positions are measured with a position vector $\vec{r}_{Xm}$. We now state that the placeholder variable $B$ is angular momentum $\vec{r}_{Xm} \wedge m\vec{V}$. It follows that $\mathrm{d}B/\mathrm{d}t$ becomes $\mathrm{d}\vec{r}_{Xm} \wedge m\vec{V}_{\text{sys}}/\mathrm{d}t$, which by definition is the net moment $\vec{M}_{\text{net}}$ applying to the system (see again eq. 1/26 p. 20). Also, $b \equiv B/m = \vec{r}_{Xm} \wedge \vec{V}$ and now the Reynolds transport theorem becomes:

$$\frac{\mathrm{d}(\vec{r}_{\mathrm{X}m} \wedge m\vec{V})_{\mathrm{sys}}}{\mathrm{d}t} = \vec{M}_{\mathrm{net,X}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\mathrm{CV}} \vec{r}_{\mathrm{X}m} \wedge \rho \vec{V} \, \mathrm{d}\mathcal{V} + \iint_{\mathrm{CS}} \vec{r}_{\mathrm{X}m} \wedge \rho \, (\vec{V}_{\mathrm{rel}} \cdot \vec{n}) \vec{V} \, \mathrm{d}A \tag{3/13}$$

| the vector sum | | the rate of change of | | the net flow of angular momentum |
|---|---|---|---|---|
| of moments | = | the angular momentum | + | through the boundaries |
| on the fluid | | within the considered volume | | of the considered volume |

in which $\vec{r}_{\mathrm{X}m}$ is a vector giving the position of any mass $m$ relative to point $X$.




Video: Making sense of the angular momentum balance equation

by Olivier Cleynen (CC-BY)
https://youtu.be/VR8LGr6PuRY

Sometimes, the control volume has well-defined inlets and outlets through which the flow $\vec{r}_{\mathrm{X}m} \wedge \rho \vec{V}(\vec{V}_{\mathrm{rel}} \cdot \vec{n})$ is uniform: this is illustrated in figure 3.5. In that case equation 3/13 reduces to a more readable form:

$$\vec{M}_{\mathrm{net,X}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\mathrm{CV}} \vec{r}_{\mathrm{X}m} \wedge \rho \vec{V} \, \mathrm{d}\mathcal{V} + \sum_{\mathrm{out}} \left\{ \vec{r}_{\mathrm{X}m} \wedge |\dot{m}| \vec{V} \right\} - \sum_{\mathrm{in}} \left\{ \vec{r}_{\mathrm{X}m} \wedge |\dot{m}| \vec{V} \right\} \tag{3/14}$$

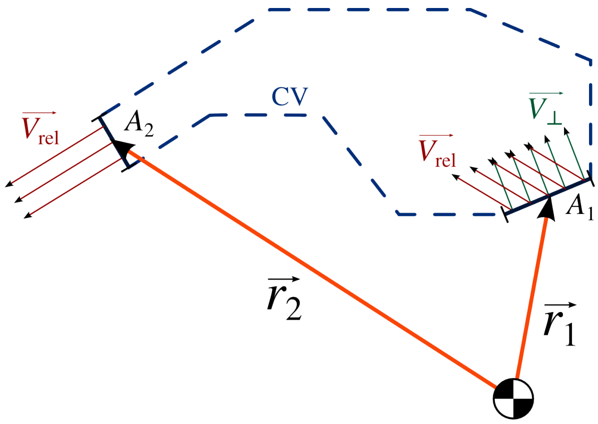



Figure 3.5: A control volume for which the properties of the system are uniform at each inlet or outlet. Here the moment about point X in the bottom right is $\vec{M}_{\mathrm{net,X}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\mathrm{CV}} \vec{r}_{\mathrm{X}m} \wedge \rho \vec{V} \, \mathrm{d}\mathcal{V} + \vec{r}_2 \wedge |\dot{m}_2| \vec{V}_2 - \vec{r}_1 \wedge |\dot{m}_1| \vec{V}_1$.

*Figure* CC-0 *Olivier Cleynen*

The same remarks we made for linear momentum earlier apply here: there are many possible phenomena which may equate to a moment on the fluid. We will explore this equation rather shyly in the problem section of the chapter.

*Advice from an expert*

The angular momentum balance equation is useful in cases where we attempt to balance a machine using fluid flows. Moments can be added and subtracted as vectors, just like forces. It's an extra layer of abstraction to learn (just about every fluid dynamicist has suffered at first with the clockwise-is-positive convention). You'll be glad you learned it when you attempt to prevent the helicopter from spinning uncontrollably — or when you ask your CFD software to calculate a moment, and it spits out a vector!

## 3.6 Balance of energy

What power is applied to the fluid for it to travel through the control volume? We can answer this question by writing an energy balance in the template provided by the Reynolds transport theorem (eq. 3/5).

The derivation of the equation we look for is the same as the derivation of equation 2/18 p. 40 in the last chapter. We may therefore directly jump to the result:

$$\frac{\mathrm{d}E_{\text{sys}}}{\mathrm{d}t} = \dot{Q}_{\text{net}} + \dot{W}_{\text{shaft, net}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho \, e_f \, \mathrm{d}\mathcal{V} + \iint_{\text{CS}} \rho \, e_f \, (\vec{V}_{\text{rel}} \cdot \vec{n}) \, \mathrm{d}A \tag{3/15}$$

with the term $e_f$ carrying all of the energy terms relevant for us in fluid mechanics:

$$e_f \equiv i + \frac{p}{\rho} + \frac{1}{2}V^2 + gz \tag{3/16}$$

It can be seen here that this equation only differs from equation 2/18 in the last chapter in the two following ways:

- There is an unsteady term (first term on the right-hand side), accounting for the accumulation or depletion of energy within the control volume;
- The energy flows through the inlet and outlet are expressed as integrals, allowing us to account for non-uniform distributions.

Save for those two differences, the equation is not any different — and, it must be admitted, not much more useful in practice. In this course, we will not be using this equation to solve problems.

## 3.7 Limits of integral analysis

Integral analysis is an incredibly useful tool in fluid dynamics: in any given problem, it allows us to rapidly describe and calculate the main fluid phenomena at hand. The net force exerted on the fluid as it is deflected downwards by a helicopter, for example, can be calculated using just a loosely-drawn control volume and a single vector equation.

As we progress through the end-of-chapter problems, however, the limits of this method slowly become apparent. There are two of them:

- First, we are confined to calculating the *net* effect of fluid flow. The net force, for example, encompasses the integral effect of all forces —due to pressure, shear, and gravity— applied on the fluid as it transits through the control volume. Integral analysis gives us absolutely no way of distinguishing between those sub-components. In order to do that (for example, to calculate which part of a pump's mechanical power is lost to internal viscous effects), we would need to look within the control volume.

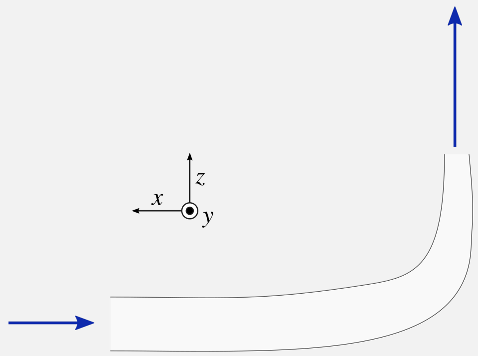
- Second, all four of our equations in this chapter only work in one direction. The value $\mathrm{d}B_{\text{sys}}/\,\mathrm{d}t$ of any finite integral cannot be used to find which function $\rho b V_\perp \,\mathrm{d}A$ was integrated over the control surface to obtain it. For example, there are an *infinite* number of velocity profiles which will result in a net force of +12 N. Knowing the net value of an integral, we cannot deduce the conditions which lead to it.
  In practice, this is a major limitation on the use of integral analysis, because it confines us to working with large swaths of experimental data gathered at the borders of our control volumes. From the wake below the helicopter, we deduce the net force; but the net force tells us nothing about the shape of the wake.

Clearly, in order to overcome these limitations, we are going to need to open up the control volume, and look at the details of the flow within — perhaps by dividing it into a myriad of sub-control volumes. This is what we set ourselves to in chapter 6 (*Prediction of fluid flows*), with a thundering and formidable methodology we shall call *derivative analysis*.

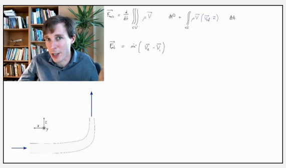
## 3.8 Solved problems

**Flow through pipe bend**

A chemical is flowing through a pipe with 90° bend, with a mass flow of 200 $\mathrm{kg\,s^{-1}}$. Its incoming velocity (uniform) is 2 $\mathrm{m\,s^{-1}}$, and its outgoing velocity is 3 $\mathrm{m\,s^{-1}}$.



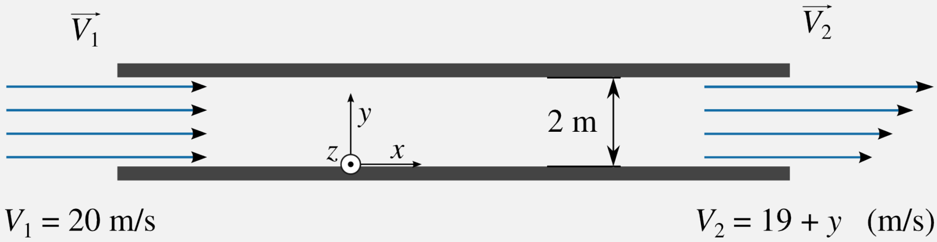

What is the net force exerting on the fluid?



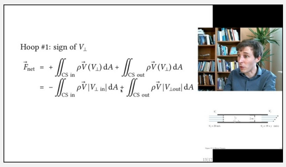
*See this solution worked out step by step on YouTube*
https://youtu.be/npEbzOx3qHY (CC-BY Olivier Cleynen)

### Outlet with a non-uniform velocity

Water is flowing through a straight rectangular pipe. At inlet, its velocity is uniform, with $V_1 = 20\,\mathrm{m\,s^{-1}}$. At the outlet, the velocity is the same on average; however, it is not uniformly distributed. We have $V_2 = 19 + y$ ($\mathrm{m\,s^{-1}}$), with $y$ the vertical coordinate (in meters). The width of the pipe in the $z$-direction is $\Delta z = 1\,\mathrm{m}$.




What is the net force exerting on the fluid?

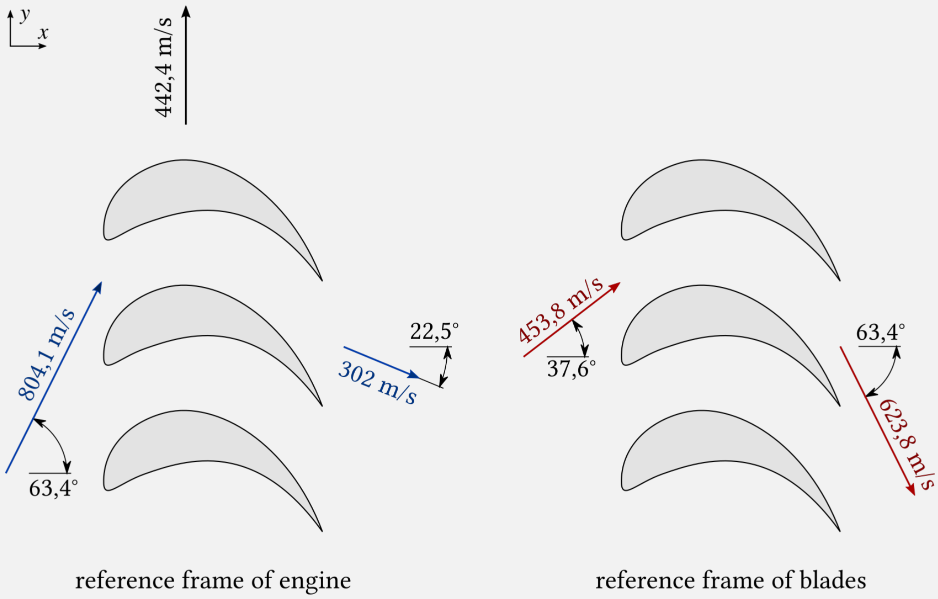


*See this solution worked out step by step on YouTube*
https://youtu.be/hPRvLVolHtY 

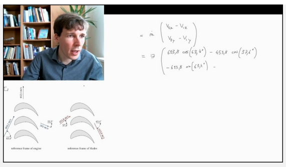
### Force and power on a moving turbine blade

In a jet engine, a row of blades is moving with great speed ($V_{\text{blade}} = 442{,}4\,\mathrm{m\,s^{-1}}$). Each blade receives $1{,}7\,\mathrm{kg\,s^{-1}}$ of high-speed, high-temperature, high-pressure gas. Through both the movement and the shape of the blade, this gas is deviated and their properties change. For simplicity, we assume the properties at inlet and outlet are uniform.

The incoming and outgoing velocities are described below two times: once (left) from the point of view of the stationary engine, and once (right) from the point of view of the moving blades.




What is the net force exerting on each blade? What is the power transmitted to the blade?



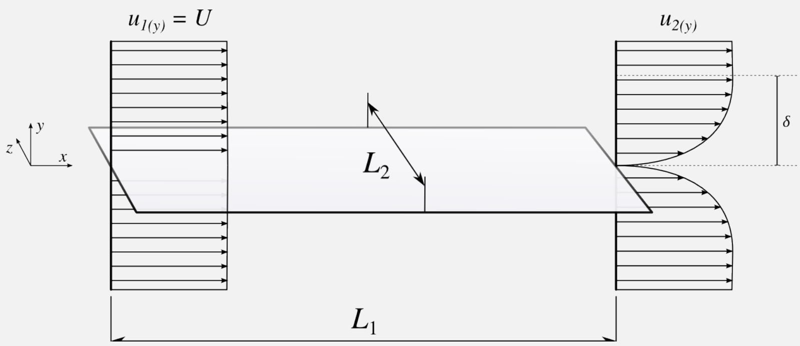
*See this solution worked out step by step on YouTube*
https://youtu.be/BgUjpaBYeDc (CC-BY Olivier Cleynen)

**Mass flows in a boundary layer**

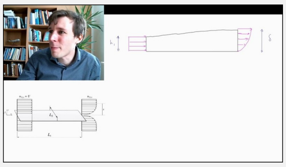
A flat plate is positioned parallel to a steady, uniform air flow ($u_1 = U = 25\,\mathrm{m\,s^{-1}}$). At the trailing edge of the plate, the velocity $u_2$ is measured. It is a function of height $y$, with thickness $\delta = 2\,\mathrm{cm}$, following the distribution:

$$u_2 = U\left(\frac{y}{\delta}\right)^{\frac{1}{6}} \tag{3/17}$$

The plate has length $L_1 = 50\,\mathrm{cm}$ (in $x$-direction, along the flow) and width $L_2 = 80\,\mathrm{cm}$ (in $z$-direction, across the flow). The density of air is $1{,}225\,\mathrm{kg\,m^{-3}}$.



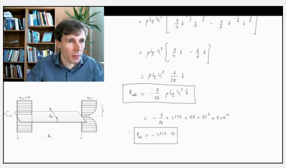

The layer has thickness $\delta$ at outlet. What is the *inlet* height of a control volume that has the mass flow of this layer?



*See this solution worked out step by step on YouTube*
https://youtu.be/2ROV07UcjiI (CC-BY Olivier Cleynen)

**Shear force in a boundary layer**

In the flat plate problem from above, what is the net force exerted on the air by the plate?



*See this solution worked out step by step on YouTube*
https://youtu.be/Lu3GRcv3BSA (CC-BY Olivier Cleynen)

# Problem sheet 3: Analysis of existing flows with three dimensions

last edited May 15, 2020

by Olivier Cleynen — https://fluidmech.ninja/

Except otherwise indicated, assume that:

The atmosphere has $p_{\text{atm.}} = 1\,\text{bar}$; $\rho_{\text{atm.}} = 1{,}225\,\text{kg}\,\text{m}^{-3}$; $T_{\text{atm.}} = 11{,}3\,°\text{C}$; $\mu_{\text{atm.}} = 1{,}5 \cdot 10^{-5}\,\text{Pa}\,\text{s}$

Air behaves as a perfect gas: $R_{\text{air}}$=287 J kg$^{-1}$ K$^{-1}$; $\gamma_{\text{air}}$=1,4; $c_{p\,\text{air}}$=1 005 J kg$^{-1}$ K$^{-1}$; $c_{v\,\text{air}}$=718 J kg$^{-1}$ K$^{-1}$

Liquid water is incompressible: $\rho_{\text{water}} = 1\,000\,\text{kg}\,\text{m}^{-3}$, $c_{p\,\text{water}} = 4\,180\,\text{J}\,\text{kg}^{-1}\,\text{K}^{-1}$

Mass balance through an arbitrary volume:

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho \,\mathrm{d}\mathcal{V} + \iint_{\text{CS}} \rho \left(\vec{V}_{\text{rel}} \cdot \vec{n}\right) \mathrm{d}A \tag{3/6}$$

Momentum balance through an arbitrary volume:

$$\vec{F}_{\text{net}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \rho \vec{V} \,\mathrm{d}\mathcal{V} + \iint_{\text{CS}} \rho \vec{V} \left(\vec{V}_{\text{rel}} \cdot \vec{n}\right) \mathrm{d}A \tag{3/9}$$

Angular momentum balance through an arbitrary volume:

$$\vec{M}_{\text{net,X}} = \frac{\mathrm{d}}{\mathrm{d}t} \iiint_{\text{CV}} \vec{r}_{\text{X}m} \wedge \rho \vec{V} \,\mathrm{d}\mathcal{V} + \iint_{\text{CS}} \vec{r}_{\text{X}m} \wedge \rho \left(\vec{V}_{\text{rel}} \cdot \vec{n}\right) \vec{V} \,\mathrm{d}A \tag{3/13}$$

## 3.1 Reading quiz

Once you are done with reading the content of this chapter, you can go take the associated quiz at https://elearning.ovgu.de/course/view.php?id=7199

In the winter semester, quizzes are not graded.

(the quiz for chapter 3 will open from May 10 to May 20)

## 3.2 Pipe bend



A pipe with diameter 30 mm has a bend with angle $\theta = 130°$, as shown in fig. 3.6. Water enters and leaves the pipe with the same speed $V_1 = V_2 = 1{,}5\,\text{m}\,\text{s}^{-1}$. The velocity distribution at both inlet and outlet is uniform.

3.2.1. What is the mass flow traveling through the pipe?

3.2.2. What is the force exerted by the pipe bend on the water?

3.2.3. Represent the force vector qualitatively (i.e. without numerical data).

3.2.4. What would be the new force if all of the speeds were doubled?

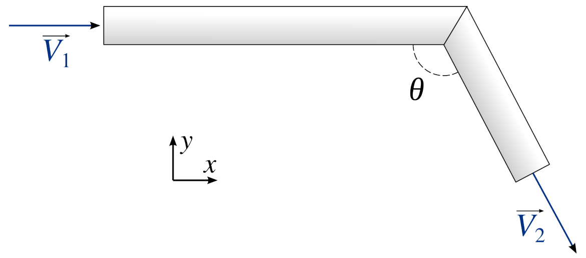



Figure 3.6: A pipe bend, through which water is flowing. We assume that the velocity distribution at inlet and outlet is identical.

*Figure* CC-0 *Olivier Cleynen*

## 3.3 Exhaust gas deflector

A deflector is used behind a stationary aircraft during ground testing of a jet engine (fig. 3.7).

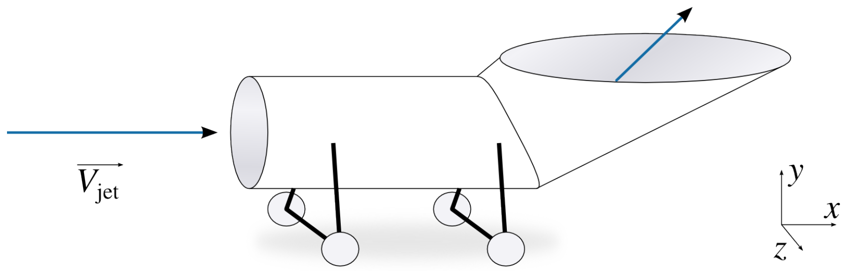



Figure 3.7: A mobile exhaust gas deflector, used to deflect hot jet engine exhaust gases upwards during ground tests.

*Figure* CC-0 *Olivier Cleynen*

The deflector is fed with a horizontal air jet with a quasi-uniform velocity profile; the speed is $V_{\text{jet}} = 600\,\text{km}\,\text{h}^{-1}$, temperature 400 °C and the pressure is atmospheric. As the exhaust gases travel through the pipe, their heat losses are negligible. Gases are rejected with a 40° angle relative to the horizontal.

The inlet diameter is 1 m and the horizontal outlet surface is 6 m$^2$.

3.3.1. What is the force exerted on the ground by the deflection of the exhaust gases?

3.3.2. Describe qualitatively (i.e. without numerical data) a modification to the deflector that would reduce the horizontal component of force.

3.3.3. What would the force be if the deflector traveled rearwards (positive $x$-direction) with a velocity of $10\,\text{m}\,\text{s}^{-1}$?

## 3.4 Pelton water turbine

*White [22] P3.56*

A water turbine is modeled as the following system: a water jet exiting a stationary nozzle hits a blade which is mounted on a rotor (fig. 3.8). In the ideal case, viscous effects can be neglected, and the water jet is deflected entirely with a 180° angle.

The nozzle has a cross-section diameter of 5 cm and produces a water jet with a speed $V_{\text{jet}} = 15\,\text{m}\,\text{s}^{-1}$. The rotor diameter is 2 m and the blade height is negligibly small.

We first study the case in which the rotor is prevented from rotating, so that the blade is stationary ($V_{\text{blade}} = 0\,\text{m}\,\text{s}^{-1}$).

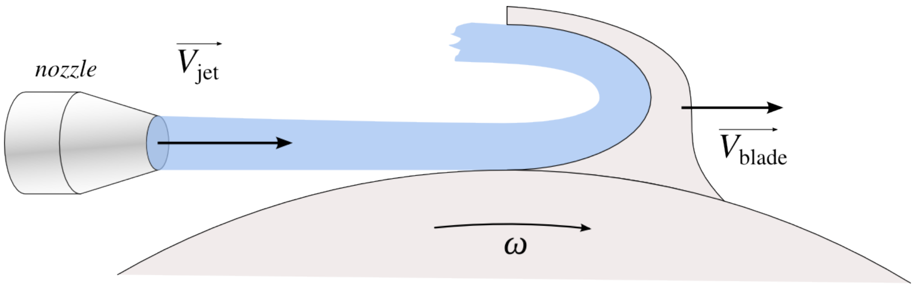



Figure 3.8: Schematic drawing of a water turbine blade. This type of turbine is called Pelton turbine.

*Figure* CC-0 *Olivier Cleynen*

3.4.1. What is the force exerted by the water on the blade?

3.4.2. What is the moment exerted by the blade around the rotor axis?

3.4.3. What is the power transmitted to the rotor?

We now let the rotor rotate freely. Friction losses are negligible, and it accelerates until it reaches maximum velocity.

3.4.4. What is the rotor rotation speed?

3.4.5. What is the power transmitted to the rotor?

The rotor is now coupled to an electrical generator.

3.4.6. Show that the maximum power that can be transmitted to the generator occurs for $V_{blade} = \frac{1}{3} V_{water}$.

3.4.7. What is the maximum power that can be transmitted to the generator?

3.4.8. How would the above result change if viscous effects were taken into account? (briefly justify your answer, e.g. in 30 words or less)

---

## 3.5 Snow plow

*derived from Gerhart & Gross [7] Ex5.9*

A road-based snow plow (fig. 3.9) is clearing up the snow on a flat surface. We wish to quantify the power required for its operation.

The snow plow is advancing at $25\,\mathrm{km\,h^{-1}}$; its blade has a frontal-view width of $4\,\mathrm{m}$.

The snow on the ground is $30\,\mathrm{cm}$ deep and has density $300\,\mathrm{kg\,m^{-3}}$.

The snow is pushed along the blade and is rejected horizontally with a $30^\circ$ angle to the left of the plow. Its density has then risen to $450\,\mathrm{kg\,m^{-3}}$. The cross-section area $A_{outlet}$ of the outflowing snow in the $x$-$y$ plane is $1{,}1\,\mathrm{m^2}$.

3.5.1. What is the force exerted on the blade by the deflection of the snow? (Indicate its magnitude and coordinates)

3.5.2. What is the power required for the operation of the snow plow?

3.5.3. If the plow velocity was increased by $10\,\%$, what would be the increase in power?

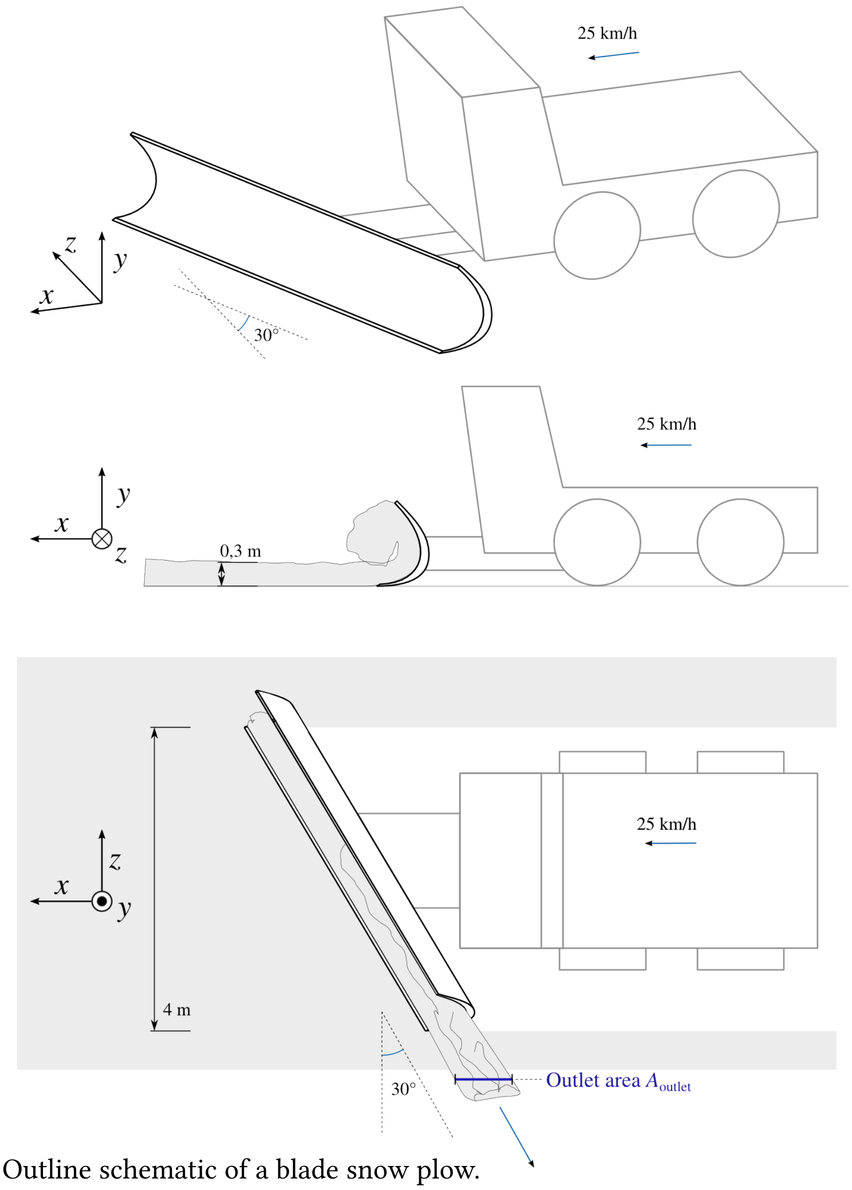



Figure 3.9: Outline schematic of a blade snow plow.



## 3.6 Inlet of a pipe

*Based on White [22]*

Water is circulated inside a cylindrical pipe with diameter 1 m (fig. 3.10).

At the entrance of the pipe, the speed is uniform: $u_1 = U_{\text{av.}} = 5\,\text{m}\,\text{s}^{-1}$.

Shear applies on the fluid from the walls, where the velocity is zero. This strains the fluid particles, and changes the velocity distribution. At the outlet of the pipe, the velocity profile is no longer uniform. It can be modeled as a function of the radius with the relationship:

$$u_{2(r)} = U_{\text{center}} \left(1 - \frac{r}{R}\right)^{\frac{1}{7}} \tag{3/18}$$

What is the center velocity $U_{\text{center}}$ at the outlet?

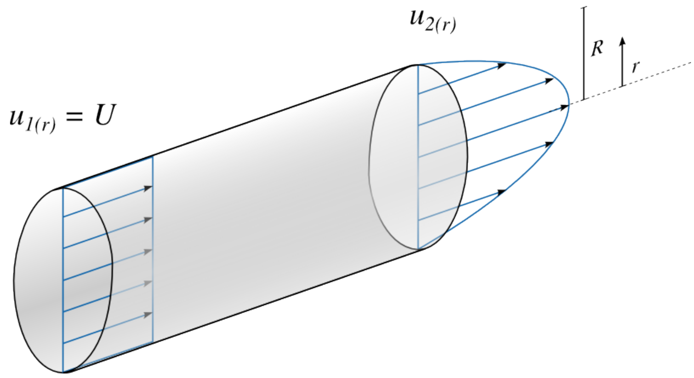



Figure 3.10: Velocity profiles at the inlet and outlet of a circular pipe.



## 3.7 Drag on a cylindrical profile

In order to measure the drag on a cylindrical profile, a cylindrical tube is positioned perpendicular to the air flow in a wind tunnel (fig. 3.11), and the longitudinal component of velocity is measured across the tunnel section.

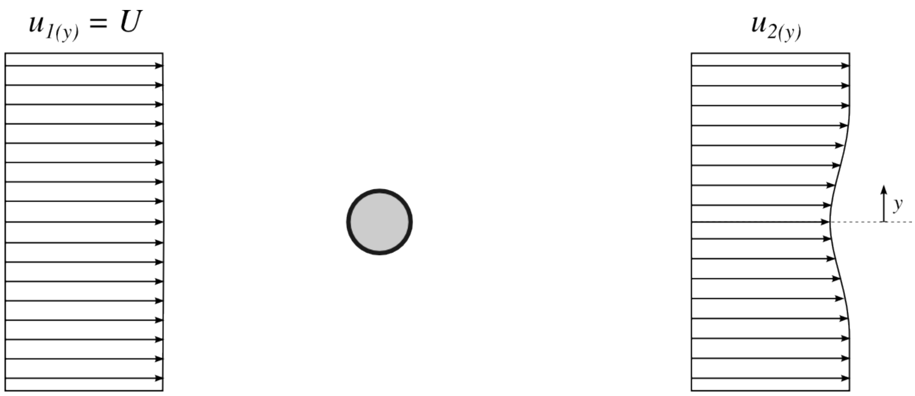



Figure 3.11: A cylinder profile set up in a wind tunnel, with the air flowing from left to right.



Upstream of the cylinder, the air flow velocity is uniform ($u_1 = U = 30\,\mathrm{m\,s^{-1}}$).

Downstream of the cylinder, the speed is measured across a 2 m height interval. Horizontal speed measurements are gathered and modeled with the following relationship:

$$u_{2(y)} = 29 + y^2 \tag{3/19}$$

The width of the cylinder (perpendicular to the flow) is 2 m. The Mach number is very low, and the air density remains constant at $\rho = 1{,}23\,\mathrm{kg\,m^{-3}}$; pressure is uniform all along the measurement field.

3.7.1. What is the drag force applying on the cylinder?

3.7.2. How would this value change if the flow in the cylinder wake was turbulent, and the function $u_{2(y)}$ above only modeled *time-averaged values* of the horizontal velocity? (briefly justify your answer, e.g. in 30 words or less)

## 3.8 Drag on a flat plate

We wish to measure the drag applying on a thin plate positioned parallel to an air stream. In order to achieve this, measurements of the horizontal velocity $u$ are made around the plate (fig. 3.12).

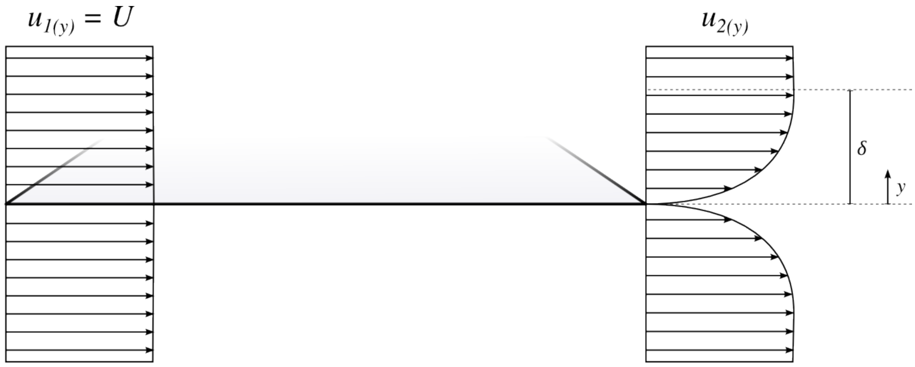



Figure 3.12: Side view of a plate positioned parallel to the flow.

*Figure* CC-0 *Olivier Cleynen*

At the leading edge of the plate, the horizontal velocity of the air is uniform: $u_1 = U = 10\,\mathrm{m\,s^{-1}}$.

At the trailing edge of the plate, we observe that a thin layer of air has been slowed down by the effect of shear. This layer, called *boundary layer*, has a thickness of $\delta = 1\,\mathrm{cm}$. The horizontal velocity profile can be modeled with the relation:

$$u_{2(y)} = U \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \tag{3/20}$$

The width of the plate (perpendicular to the flow) is $30\,\mathrm{cm}$ and it has negligible thickness. The flow is incompressible ($\rho = 1{,}23\,\mathrm{kg\,m^{-3}}$) and the pressure is uniform.

3.8.1. What is the drag force applying on the plate?

3.8.2. What is the power required to compensate the drag?

3.8.3. Under which form is the kinetic energy lost by the flow carried away? Can this new form of energy be measured? (briefly justify your answer, e.g. in 30 words or less)

## 3.9 Drag measurements in a wind tunnel

A group of students proceeds with speed measurements in a wind tunnel. The objective is to measure the drag applying on a wing profile positioned across the tunnel test section (fig. 3.13).

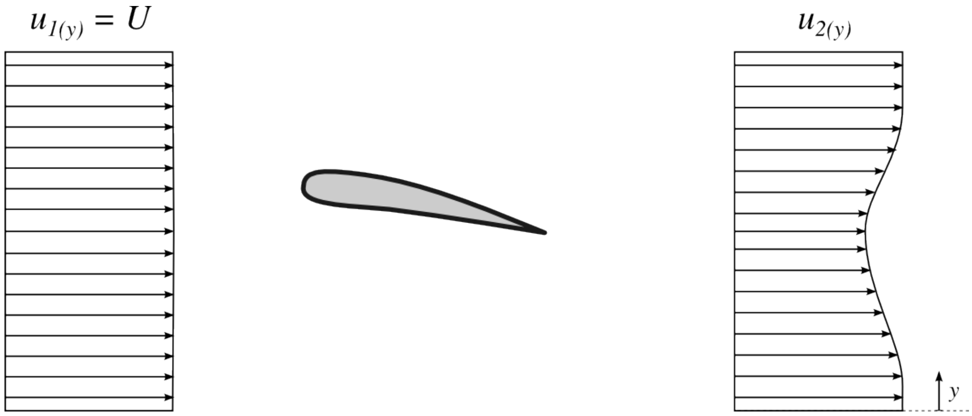



Figure 3.13: Wing profile positioned across a wind tunnel. The horizontal velocity distributions upstream and downstream of the profile are also shown.

*Figure* CC-0 *Olivier Cleynen*

Upstream of the profile, the air flow velocity is uniform ($u_1 = U = 50\,\mathrm{m\,s^{-1}}$).

Downstream of the profile, horizontal velocity measurements are made every 5 cm across the flow; the following results are obtained:

| vertical position (cm) | horizontal speed $u_2$ ($\mathrm{m\,s^{-1}}$) |
|---|---|
| 0 | 50 |
| 5 | 50 |
| 10 | 49 |
| 15 | 48 |
| 20 | 45 |
| 25 | 41 |
| 30 | 39 |
| 35 | 40 |
| 40 | 43 |
| 45 | 47 |
| 50 | 48 |
| 55 | 50 |
| 60 | 50 |

The width of the profile (perpendicular to the flow) is 50 cm. The airflow is incompressible ($\rho = 1{,}23\,\mathrm{kg\,m^{-3}}$) and the pressure is uniform across the measurement surface.

3.9.1. What is the drag applying on the profile?

3.9.2. How would the above calculation change if *vertical* speed measurements were also taken into account?

## 3.10 Moment on gas deflector

*non-examniable*

We revisit the exhaust gas deflector of exercise 3.3 p. 64. Figure 3.14 below shows the deflector viewed from the side. The midpoint of the inlet is 2 m above and 5 m behind the wheel labeled "**A**", while the center axis of the outlet passes 1,72 m away from it, as represented in fig 3.14

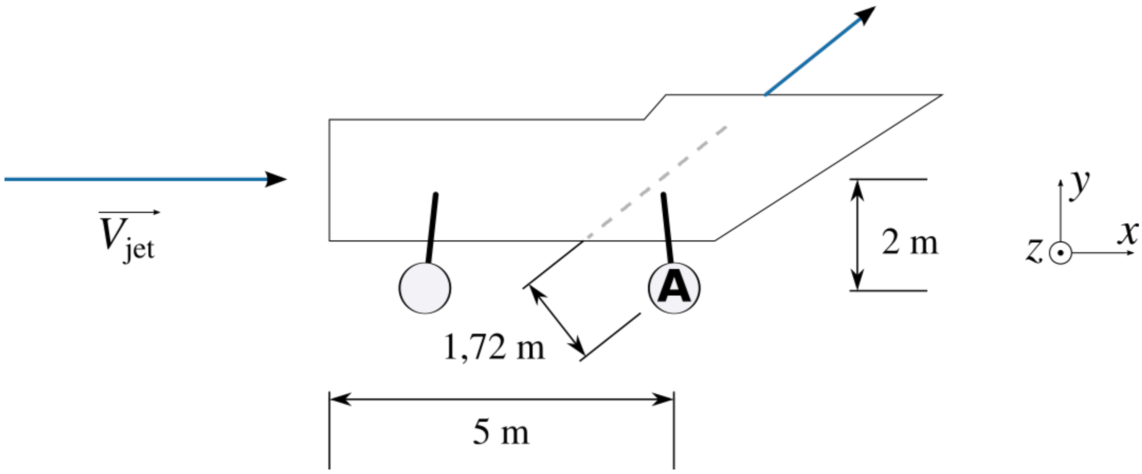



Figure 3.14: Side view of the mobile exhaust gas deflector which was shown in fig. 3.7 p. 64



What is the moment generated by the gas flow about the axis of the wheel labeled "**A**"?

## 3.11 Helicopter tail moment

*non-examinable*

In a helicopter, the role of the tail is to counter exactly the moment exerted by the main rotor about the main rotor axis. This is usually done using a tail rotor which is rotating around a horizontal axis.

A helicopter, shown in figure 3.15, is designed to use a tail *without* a rotor, so as to reduce risks of accidents when landing and taking off. The tail is a long hollow cylindrical tube with two inlets and one outlet. To simplify calculations, we consider that pressure is atmospheric at every inlet and outlet.

- inlet A has a cross-section area of 0,2 m$^2$. It contributes hot exhaust gases of density 0,8 kg m$^{-3}$ and velocity 12 m s$^{-1}$, aligned with the ($x$) axis of the tail;
- inlet B contributes 25 kg s$^{-1}$ of atmospheric air incoming at an angle $\alpha = 130°$ relative to the axis of the tail, with a velocity of 3 m s$^{-1}$.

The mix of exhaust gases and atmospheric air is rejected at the tip of the tail (outlet C) with a fixed velocity of 45 m s$^{-1}$. The angle $\theta$ at which gases are rejected is controlled by the flight computer.

3.11.1. What is the rejection angle $\theta$ required so that the tail generates a moment of +6 kN m around the main rotor ($y$) axis?

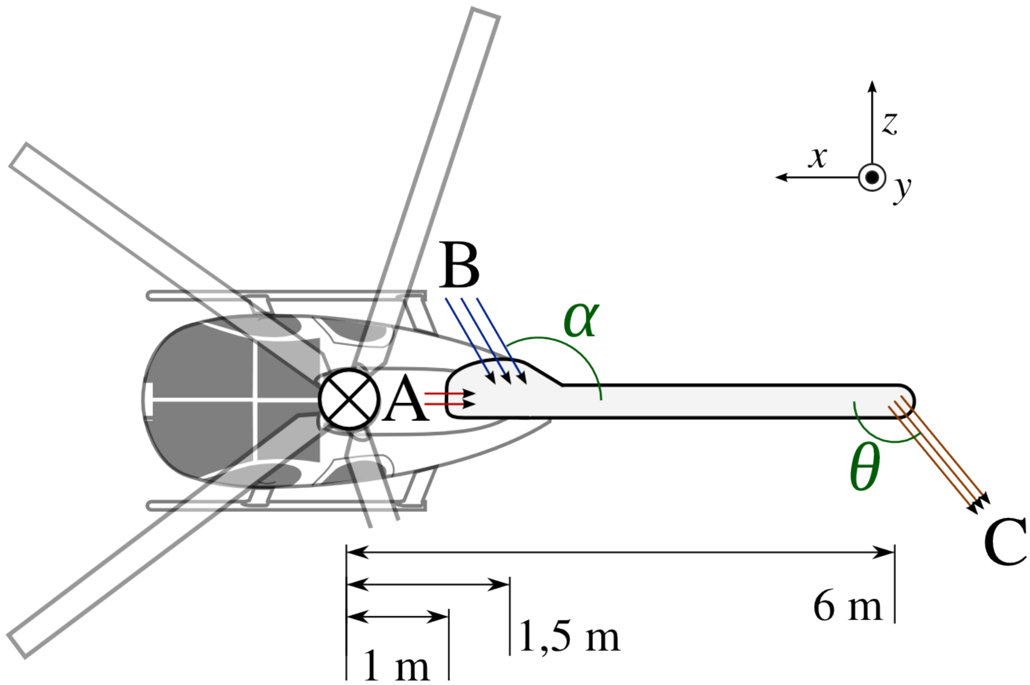



Figure 3.15: Top-view of a helicopter using a rotor-less tail.

*Figure derived from a figure* CC-BY-SA *by Commons User:FOX 52*

3.11.2. Propose and quantify a modification to the tail geometry or operating conditions that would allow the tail to produce no thrust (that is to say, zero force in the $x$-axis), while still generating the same moment.

*Remark: this system is commercialized by MD Helicopters as the* NOTAR. *The use of exhaust gases was abandoned, however, a clever use of air circulation around the tail pipe axis contributes to the generated moment; this effect is explored in chapter 11 (Large- and small-scale flows) (§11.3.4 p. 228).*

# Answers

**3.2** 1) $\dot{m} = 1{,}0603\,\mathrm{kg\,s^{-1}}$
2) $\vec{F}_{\mathrm{net}} = (-0{,}5681; -1{,}2184)\,\mathrm{N}$
4) The force is quadrupled.

**3.3** 1) $\vec{F}_{\mathrm{net}} = (-9{,}532; +1{,}479)\,\mathrm{kN} : ||\vec{F}_{\mathrm{net}}|| = 9{,}646\,\mathrm{kN}$ (force on ground is opposite: $\vec{F}_{\text{fluid on pipe}} = -\vec{F}_{\text{pipe on fluid}}$);
3) $||\vec{F}_{\mathrm{net}\,2}|| = 8{,}525\,\mathrm{kN}$.

**3.4** 1) $F_{\mathrm{net}} = (-883{,}6; 0)\,\mathrm{N}$;
2) $M_{\mathrm{net}\,X} = |F_{\mathrm{net}}|R = 883{,}6\,\mathrm{N\,m}$;
3) $\dot{W}_{\mathrm{rotor}} = 0\,\mathrm{W}$;
4) $\omega = 143{,}2\,\mathrm{rpm}$ ($F_{\mathrm{net}} = 0\,\mathrm{N}$);
5) $\dot{W}_{\mathrm{rotor}} = 0\,\mathrm{W}$ again;
6) $\dot{W}_{\text{rotor, max}} = 1{,}963\,\mathrm{kW}$ @ $V_{\text{blade, optimal}} = \frac{1}{3} V_{\text{water jet}}$.

**3.5** 1) $\dot{m} = 2\,500\,\mathrm{kg\,s^{-1}}$; $F_{\mathrm{net}\;x} = +10{,}07\,\mathrm{kN}$, $F_{\mathrm{net}\;z} = -12{,}63\,\mathrm{kN}$ (force on blade is opposite);
2) $\dot{W} = \vec{F}_{\mathrm{net}} \cdot \vec{V}_{\mathrm{plow}} = F_{\mathrm{net}x}|V_1| = 69{,}94\,\mathrm{kW}$
3) $\dot{W}_2 = 1{,}1^3\,\dot{W}$ (+33 %)

**3.6** $V_{\mathrm{center}} = 1{,}2245U$

**3.7** $F_{\mathrm{net}x} = \rho L \int_{S2} \left(u_2^2 - U u_2\right)\,\mathrm{d}y = -95{,}78\,\mathrm{N}$.

**3.8** $F_{\mathrm{net}x} = \rho L \int_0^{\delta} \left(u_{(y)}^2 - U u_{(y)}\right)\,\mathrm{d}y = -7{,}175 \cdot 10^{-2}\,\mathrm{N} : \dot{W}_{\mathrm{drag}} = U|F_{\mathrm{net}x}| = 0{,}718\,\mathrm{W}$.

**3.9** $F_{\mathrm{net}x} \approx \rho L \Sigma_y \left[\left(u_2^2 - U u_2\right)\,\delta y\right] = -64{,}8\,\mathrm{N}$.

**3.10** Re-use $\dot{m} = 67{,}76\,\mathrm{kg\,s^{-1}}$, $V_1 = 166{,}7\,\mathrm{m\,s^{-1}}$, $V_2 = 33{,}94\,\mathrm{m\,s^{-1}}$ from ex. 3.3. With $R_{2\perp V_2} = 1{,}717\,\mathrm{m}$, plug in numbers in eq. 3/14 p. 58: $M_{\mathrm{net}} = +18{,}64\,\mathrm{kN\,m}$ in $z$-direction.

**3.11** 1) Work eq. 3/13 down to scalar equation (in $y$-direction), solve for $\theta$: $\theta = 123{,}1°$.
2) There are multiple solutions which allow both moment and force equations to be solved at the same time. $r_{\mathrm{C}}$ can be shortened, the flow in C can be split into forward and rearward components, or tilted downwards etc. Reductions in $\dot{m}_{\mathrm{B}}$ or $V_{\mathrm{C}}$ are also possible, but quantifying them requires solving both equations at once.