

Fluid Dynamics

Chapter 2 – Analysis of existing flows with one dimension

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by Olivier Cleynen – <https://fluidmech.ninja/>

2.1 Motivation	33
2.2 One-dimensional flow problems	33
2.3 Balance of mass	35
2.3.1 Mass balance equation	35
2.3.2 Problems with the mass balance equation	36
2.4 Balance of momentum	37
2.5 Balance of energy	39
2.6 The Bernoulli equation	41
2.6.1 Theory	41
2.6.2 Reality	42
2.7 Solved problems	44
2.8 Problems	45

These notes are based on textbooks by White [22], Çengel & al.[25], Munson & al.[29], and de Nevers [17].

2.1 Motivation

In this chapter, we learn to analyze fluid flows for which a lot of information is already available. We want, when confronted to a simple flow (for example, flow entering and leaving a machine), to be able to answer three questions:

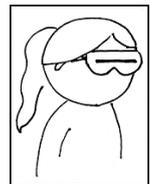
- What is the mass flow in each inlet and outlet?
- What is the force required to move the flow?
- What energy transfers are required for this movement?

2.2 One-dimensional flow problems

The method we develop here is called *integral analysis*, because it involves calculating the overall (integral) effect of the fluid flowing through a considered volume. In this chapter, we consider one-dimensional flows (at least in a loose definition); we will consider more advanced cases in chapter 3 (*Analysis of existing flows with three dimensions*).

For now, we are interested in flows where four conditions are met:

1. There is a clearly identified inlet and outlet;
2. At inlet and outlet, the fluid properties are uniform, so that they can be evaluated in bulk (e.g. the inlet has only one velocity, one temperature etc.);
3. There are no significant changes in flow direction;



Abstruse Goose #237: how many
dimensions are enough for you?
by an anonymous artist (CC-BY-NC)
<https://abstrusegoose.com/237>

4. A lot of information is available about the fluid properties at inlet.

Providing that those conditions are met, we can answer the question: what is the *net* effect of the fluid flow through the considered volume?

In order to write useful equations, we need to begin with rigorous definitions, with the help of figure 2.1:

- We call *control volume* a certain volume we are interested in. Fluid flows through the control volume.
In this chapter, the control volume does not change with time. The fluid flow does not change in time, either (i.e. the flow is steady). The fluid enters and leaves the control volume at a clearly-identifiable inlet and outlet.
- At a certain instant, the mass of fluid that is inside the control volume is called the *system*. The system is traveling. At a later point in time, it has moved and deformed.

The three equations that we write in this chapter state that basic physical laws apply to the system. They are *balance equations* (see §1.7 p. 20). Each time, we will express **what is happening to the system, as a function of the fluid properties at the inlet and outlet of the control volume**. This will allow us to answer three questions:

- What is the mass flow entering and leaving the control volume?
- What is the force required to move the flow through the control volume?
- What energy transfer is required to move the flow through the control volume?

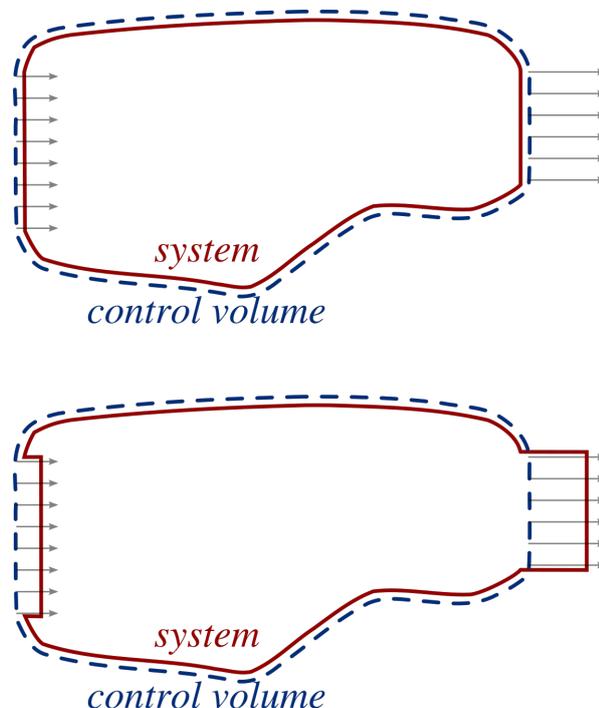


Figure 2.1: A control volume within a flow. The *system* is the amount of mass included within the control volume at a given time. Because mass enters and leaves the control volume, the system is being moved and deformed (bottom).

Figure CC-0 Olivier Cleynen

2.3 Balance of mass

2.3.1 Mass balance equation

How much mass flow is coming in and out of the control volume? We answer this question by writing a mass balance equation. It compares the rate of change of the system's mass (which by definition is zero, see eq. 1/24 p. 20), to the flow of mass through the borders of the control volume:

$$\frac{dm_{\text{sys}}}{dt} = 0 = \frac{d}{dt} m_{\text{CV}} + \dot{m}_{\text{net}} \quad (2/1)$$

the rate of change of
the rate of change
the net mass flow
the fluid's mass
of mass inside
+
at the borders
as it transits
the considered volume
of the considered volume

Since we are here interested only in steady flows, $dm_{\text{CV}}/dt = 0$. Furthermore, we have clearly-identified inlets and outlets allowing us to re-express the net mass flow \dot{m}_{net} . Equation 2/1 becomes:

$$0 = \sum \dot{m}_{\text{incoming}} + \sum \dot{m}_{\text{outgoing}} \quad (2/2)$$

time rate of creation
the sum of
the sum of
or destruction of mass
incoming mass flows
+
outgoing mass flows
(negative terms)
(positive terms)

For steady flow through a fixed considered volume.

The sign convention is counter-intuitive: mass flows are negative inwards and positive outwards. For example, in a case where there were two inlets and two outlets, we could write:

$$0 = \dot{m}_{\text{in } 1} + \dot{m}_{\text{in } 2} + \dot{m}_{\text{out } 1} + \dot{m}_{\text{out } 2}$$

$$0 = -|\dot{m}|_{\text{in } 1} - |\dot{m}|_{\text{in } 2} + |\dot{m}|_{\text{out } 1} + |\dot{m}|_{\text{out } 2}$$

We can substitute $\dot{m} = \rho V_{\perp} A$ (eq. 1/16 p. 19) into the equations above, obtaining:

$$0 = \sum [\rho V_{\perp} A]_{\text{incoming}} + \sum [\rho V_{\perp} A]_{\text{outgoing}} \quad (2/3)$$

For steady flow through a fixed considered volume.

Looking again at an example case where there were two inlets and two outlets, this equation 2/3 would become:

$$0 = \rho_{\text{in } 1} V_{\perp \text{ in } 1} A_{\text{in } 1} + \rho_{\text{in } 2} V_{\perp \text{ in } 2} A_{\text{in } 2} + \rho_{\text{out } 1} V_{\perp \text{ out } 1} A_{\text{out } 1} + \rho_{\text{out } 2} V_{\perp \text{ out } 2} A_{\text{out } 2}$$

$$0 = (\rho V_{\perp} A)_{\text{in } 1} + (\rho V_{\perp} A)_{\text{in } 2} + (\rho V_{\perp} A)_{\text{out } 1} + (\rho V_{\perp} A)_{\text{out } 2}$$

$$0 = -(\rho |V_{\perp}| A)_{\text{in } 1} - (\rho |V_{\perp}| A)_{\text{in } 2} + (\rho |V_{\perp}| A)_{\text{out } 1} + (\rho |V_{\perp}| A)_{\text{out } 2}$$

For steady flow through a fixed considered volume with two inlets and two outlets.

In a simple case where there is only one inlet and one outlet, this last equation can be rewritten as

$$(\rho |V_{\perp}| A)_1 = (\rho |V_{\perp}| A)_2 \quad (2/4)$$

For steady flow through a fixed considered volume with one inlet and one outlet.

2.3.2 Problems with the mass balance equation



Video: Dangers associated with the mass balance equation
by Olivier Cleynen (CC-BY)
<https://youtu.be/M1Dw6qU-FOU>

The equation 2/4 above is interesting, but also treacherous. The best way to mis-use this equation is to draw the conclusion that “if A decreases, then V must increase”. This is only true some of the time, and here are two reasons why:

1. The density ρ may change between inlet and outlet. In low-speed flows without heat transfer, ρ does not vary significantly (see §1.8 p. 1.8). But in compressible flows (for example when combustion is involved, or when compressed air is expanded), ρ may vary together with A and V . Typically, in supersonic flows (where $[Ma] > 1$, and the fluid moves faster than the speed of sound), increases in A lead to *increases* in V , because of a decrease in ρ .
2. There is no causal relationship in equation 2/4. In an incompressible flow, it may well be that reducing A_2 leads to an increase in V_2 , but nothing guarantees that the product of the two remains constant. In other words, reducing A_2 may both increase V_2 *and* decrease \dot{m} . Increases in velocity are not “for free”: they require force be applied and energy be spent. The mass balance equation cannot account for those phenomena.

Advice from an expert

Remember the title of the chapter: the tools here are for analyzing *existing* flows: those for which we can, if needed, gather more information by making measurements. If you find yourself *predicting* velocity in a machine you design with just a mass balance equation, then you might quickly find yourself making unrealistic assumptions. Immediately check what force and power are required to generate this velocity. For this, you need a momentum balance equation, and an energy balance equation.



2.4 Balance of momentum

What force is applied to the fluid for it to travel through the control volume? We answer this question by writing a momentum balance equation. It compares the rate of change of the system's momentum (which by definition is the net force applying to it, see eq. 1/25 p. 20), to the flow of momentum through the borders of the control volume:

$$\frac{d(m\vec{V}_{\text{sys}})}{dt} = \vec{F}_{\text{net}} = \frac{d}{dt} (m\vec{V})_{\text{CV}} + (\dot{m}\vec{V})_{\text{net}} \quad (2/5)$$

the rate of change of the fluid's momentum as it transits = \vec{F}_{net} = the rate of change of momentum within the considered volume + the net flow of momentum through the boundaries of the considered volume

Since we are here interested only in steady flows, and we have clearly-identified inlets and outlets, this becomes:

$$\vec{F}_{\text{net}} = \Sigma (\dot{m}\vec{V})_{\text{incoming}} + \Sigma (\dot{m}\vec{V})_{\text{outgoing}} \quad (2/6)$$

the vector sum of forces on the fluid = the sum of incoming momentum flows (with negative \dot{m} terms) + the sum of outgoing momentum flows (with positive \dot{m} terms)

The same convention as above is applied for the sign of the mass flow \dot{m} . For example, in a case where there were one inlet and one outlet, we would write:

$$\begin{aligned} \vec{F}_{\text{net}} &= (\dot{m}\vec{V})_{\text{in}} + (\dot{m}\vec{V})_{\text{out}} \\ \vec{F}_{\text{net}} &= -(|\dot{m}|\vec{V})_{\text{in}} + (|\dot{m}|\vec{V})_{\text{out}} \end{aligned}$$

For steady flow through a fixed considered volume with one inlet and one outlet.

As before, we can substitute $\dot{m} = \rho V_{\perp} A$ (eq. 1/16 p. 19) into the equations above, obtaining:

$$\vec{F}_{\text{net on fluid}} = \Sigma [\rho V_{\perp} A \vec{V}]_{\text{incoming}} + \Sigma [\rho V_{\perp} A \vec{V}]_{\text{outgoing}} \quad (2/7)$$

For steady flow through a fixed considered volume, where V_{\perp} is negative inwards, positive outwards.

In the example case where there is one inlet and one outlet, we would write:

$$\begin{aligned} \vec{F}_{\text{net}} &= (\rho V_{\perp} A \vec{V})_{\text{in}} + (\rho V_{\perp} A \vec{V})_{\text{out}} \\ \vec{F}_{\text{net}} &= -(\rho |V_{\perp}| A \vec{V})_{\text{in}} + (\rho |V_{\perp}| A \vec{V})_{\text{out}} \end{aligned}$$

To make clear a few things, let us focus on the simple case where a considered volume is traversed by a steady flow with mass flow \dot{m} , with one inlet (point 1) and one outlet (point 2). The net force \vec{F}_{net} applying on the fluid is

$$\vec{F}_{\text{net}} = |\dot{m}| (\vec{V}_2 - \vec{V}_1) \quad (2/8)$$



Video: The net force thing in the momentum balance equation
by Olivier Cleynen (CC-BY)
<https://youtu.be/AOcGNeY9ad0>

Three remarks can be made about this equation. First, we need to be aware that this is not one, but *three* equations, one for each dimension. In order to express \vec{F}_{net} , we need to calculate its three components:

$$\begin{cases} F_{\text{net}x} &= |\dot{m}|(V_{2x} - V_{1x}) \\ F_{\text{net}y} &= |\dot{m}|(V_{2y} - V_{1y}) \\ F_{\text{net}z} &= |\dot{m}|(V_{2z} - V_{1z}) \end{cases} \quad (2/9)$$

Second, there are *two* reasons why we could calculate a non-zero net force on the fluid in equation 2/8, as illustrated in fig. 2.2.

1. Even if \vec{V}_2 is aligned and in the same direction as \vec{V}_1 , they can be of different magnitude. A force is required to accelerate or decelerate the fluid (more precisely, an acceleration or deceleration of the fluid is equivalent to a force);
2. Even if \vec{V}_2 has the same magnitude as \vec{V}_1 , they can have different directions. A force is required to change the direction in which a flow is flowing (or more precisely, a change of direction is equivalent to a force).

We will explore these phenomena in greater detail in chapter 3 (*Analysis of existing flows with three dimensions*). In this current chapter, we are mostly interested in one-dimensional flows, and it will suffice for us to solve equation 2/8 in one suitable direction only, for example,

$$F_{\text{net}x} = |\dot{m}|(V_{2x} - V_{1x}) \quad (2/10)$$

The final remark is that the equation does not describe a cause-effect relationship. The net force does not cause the change in velocity any more than the change in velocity causes the net force: they are both equivalent and simultaneous. Similarly, we have no way to know what \vec{F}_{net} is made of. The exact mechanism which adds up to a net force (pressure change, shear applied through a static wall, the movement of a turbine, etc.) is “hidden” in the control volume, and unknown to us. In order to find out what happens in the control volume, we need a different type of analysis, which we will approach in chapter 6 (*Prediction of fluid flows*).

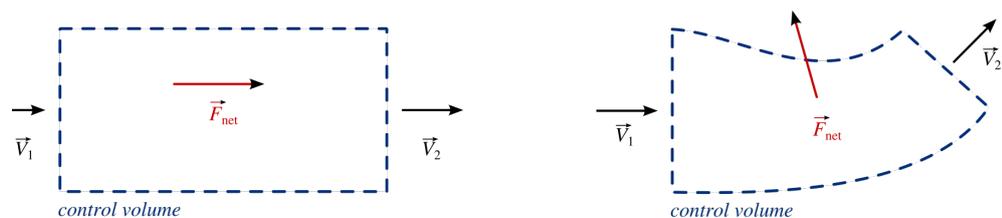


Figure 2.2: Two reasons can explain why a net force \vec{F}_{net} appears in eq. 2/8. On the left, \vec{V}_2 and \vec{V}_1 are aligned, but have different lengths. On the right \vec{V}_2 and \vec{V}_1 have the same length, but different directions. We will look at the second case in chapter 3 (*Analysis of existing flows with three dimensions*).

2.5 Balance of energy

What power is applied to the fluid for it to travel through the control volume? We answer this question by writing an energy balance equation. It compares the rate of change of the system's energy, to the flow of energy through the borders of the control volume.

For this we prefer to express the energy E as the specific energy e (in J kg^{-1}) multiplied by the mass m (kg). The time rate change of me is measured in watts ($1 \text{ W} \equiv 1 \text{ J s}^{-1}$). The energy balance equation is then:

$$\frac{dme_{\text{sys}}}{dt} = \Sigma (\dot{Q} + \dot{W}) = \frac{d}{dt} (me)_{\text{CV}} + (\dot{m}e)_{\text{net}} \quad (2/11)$$

the rate of change of the fluid's energy as it transits	=	the sum of powers as heat and work	=	the rate of change of energy within the considered volume	+	the net flow of energy through the boundaries of the considered volume
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where \dot{Q} is the power transferred as heat (W)
 \dot{W} is the power transferred as work (W)

Let us examine the terms of this equation.

The sum of powers as heat and work can be broken down as three components:

- the net power transferred as heat \dot{Q}_{net} (positive inwards);
- the net power transferred as work with moving solid surfaces $\dot{W}_{\text{surfaces, net}}$ (for example, a moving piston, turbine blade, or rotating shaft, positive inwards);
- and the net power transferred to and from the fluid *by the fluid itself*, in order to enter and leave the considered volume. This power is called *power to cross a surface* (see §1.6 p. 1.6); for each inlet or outlet we have $\dot{W}_{\text{pressure}} = -\dot{m}(p/\rho)$.

We can thus write:

$$\Sigma (\dot{Q} + \dot{W}) = \dot{Q}_{\text{net}} + \dot{W}_{\text{shaft, net}} + \dot{W}_{\text{pressure}} \quad (2/12)$$

$$= \dot{Q}_{\text{net}} + \dot{W}_{\text{shaft, net}} - \left(\dot{m} \frac{p}{\rho} \right)_{\text{net}} \quad (2/13)$$

Turning now to the specific energy e , we break it down into three components (see also §1.3.3 p. 13):

- the specific internal energy i , which represents the energy per unit mass contained as stored heat within the fluid itself. In thermodynamics, this is often noted u , but in fluid mechanics we reserve this symbol to note the x -component of velocity. In a perfect gas, i is simply proportional to absolute temperature ($i = c_v T$), but for other fluids such as water, it cannot be easily measured, and precomputed tables relating i to other properties must be used;
- the specific kinetic energy e_k ,

$$e_k \equiv \frac{1}{2} V^2 \quad (2/14)$$

- the specific potential energy e_p , related to gravity g (m s^{-2}) and altitude z (m) as:

$$e_p \equiv gz \quad (2/15)$$

We thus write out specific energy e (in J kg^{-1}) as:

$$e \equiv i + e_k + e_p \quad (2/16)$$

Now, we focus on steady flows (for which energy in the control volume does not change with time), and we can come back to eq. 2/11 to rewrite it as:

$$\begin{aligned} \dot{Q}_{\text{net}} + \dot{W}_{\text{shaft, net}} + \dot{W}_{\text{pressure}} &= (\dot{m} e)_{\text{net}} \\ \dot{Q}_{\text{net}} + \dot{W}_{\text{shaft, net}} &= (\dot{m} e)_{\text{net}} + \left(\dot{m} \frac{p}{\rho} \right)_{\text{net}} \\ \dot{Q}_{\text{net}} + \dot{W}_{\text{shaft, net}} &= \left[\dot{m} \left(i + \frac{p}{\rho} + \frac{1}{2} V^2 + gz \right) \right]_{\text{net}} \end{aligned} \quad (2/17)$$

Rewriting this into one general, usable form, we obtain:

$$\begin{aligned} \dot{Q}_{\text{net}} + \dot{W}_{\text{shaft, net}} &= \Sigma \left[\dot{m} \left(i + \frac{p}{\rho} + \frac{1}{2} V^2 + gz \right) \right]_{\text{in}} \\ &\quad + \Sigma \left[\dot{m} \left(i + \frac{p}{\rho} + \frac{1}{2} V^2 + gz \right) \right]_{\text{out}} \end{aligned} \quad (2/18)$$

For steady flow through a fixed considered volume,
where $\dot{m} = \rho V_{\perp} A$ is negative inwards, positive outwards.

This equation 2/18 is known in thermodynamics as the *steady flow energy equation* (in thermodynamics, it is usually expressed with the help of the concept of *enthalpy* $h \equiv i + p/\rho$, which we do not use here).

As usual, let us focus on a case where there is only one inlet and one outlet. We obtain:

$$\begin{aligned} \dot{Q}_{\text{net}} + \dot{W}_{\text{shaft, net}} &= - \left[|\dot{m}| \left(i + \frac{p}{\rho} + \frac{1}{2} V^2 + gz \right) \right]_{\text{in}} + \left[|\dot{m}| \left(i + \frac{p}{\rho} + \frac{1}{2} V^2 + gz \right) \right]_{\text{out}} \\ \dot{Q}_{\text{net}} + \dot{W}_{\text{shaft, net}} &= |\dot{m}| \left[\Delta i + \Delta \frac{p}{\rho} + \Delta \left(\frac{1}{2} V^2 \right) + \Delta(gz) \right] \end{aligned} \quad (2/19)$$

This equation is very useful in principle, but not so much in practice, for two reasons:

1. It contains a lot of terms. There are five fluid properties at inlet and outlet which affect energy, and it is difficult to predict which one will be affected by a heat or work transfer. For example, consider a simple water pump with known powers \dot{Q}_{in} and $\dot{W}_{\text{shaft, in}}$. An efficient pump will generate large increases in p (or V and z), while an inefficient pump will generate large increases in i and $1/\rho$. The energy balance equation, in this form, tells us nothing about how energy input to the control volume is redistributed.

Advice from an expert

Again, remember the title of the chapter. To calculate the value of any one property in equation 2/18, you need to input the value of the *eleven* other ones. It is tempting to take shortcuts while doing so (“oh, the pressure is probably the same”), with disastrous consequences. There is no solution to this. If you are attempting to predict fluid flow, and are missing information, better stop without a result than take hazardous attempts at using equation 2/18.



2. The terms have disproportionate values in practice. The heat capacity of ordinary fluids is very large, and so i is usually hundreds of times larger than the four terms in the brackets of equation 2/18. In water for example, an increase of temperature of $0,1\text{ }^\circ\text{C}$ (with the term Δi) requires the same energy as increasing its velocity from 30 km h^{-1} to 110 km h^{-1} (with the term Δe_k). This is not an issue in thermodynamics, where heat, work and temperature are the most important parameters. But in fluid mechanics, velocity is of great interest, and the energy balance is not always useful to predict its changes.

Advice from an expert

In fluid dynamics, fluid movement usually involves relatively small amounts of energy. You can convince yourself of this by dropping milk into a bowl of water: minuscule amounts of potential energy as $\Delta(gz)$ are converted into an incredibly complex distribution of velocities, before slowly dissipating into internal energy as Δi .



While *in principle*, we could calculate pressure drops or velocity changes by measuring temperature differences (and thus Δi), in practice this only works when very high powers are involved, such as in a compressor or in a rocket engine nozzle. For ordinary flow (say, air flow around a car, or water flow in a pipe), the temperature changes are much too small to be measured. See exercise 2.3 p. 46 for an example of this.

2.6 The Bernoulli equation

2.6.1 Theory

The Bernoulli equation is the energy equation applied to specific cases.

To derive the Bernoulli equation, we will start from equation 2/18 and add five constraints:

1. Steady flow.
(We had already implemented this restriction, when we set $\frac{d(me)_{cv}}{dt}$ from eq. 2/11 to zero in order to obtain eq. 2/18.)

2. Incompressible flow.
Thus, ρ stays constant;
3. No heat or work transfer.
Thus, both \dot{Q}_{net} and $\dot{W}_{\text{shaft, net}}$ are zero;
4. No friction.
Thus, the fluid internal energy i cannot increase;
5. One-dimensional flow.
Thus, our considered volume has only one inlet (labeled 1) and one outlet (labeled 2): all fluid particles move together with the same transit time, and the overall trajectory is already known.

With these five restrictions, equation 2/18 simply becomes:

$$0 + 0 = \left[\dot{m} \left(i_{\text{cst.}} + \frac{p}{\rho_{\text{cst.}}} + \frac{1}{2} V^2 + gz \right) \right]_1 + \left[\dot{m} \left(i_{\text{cst.}} + \frac{p}{\rho_{\text{cst.}}} + \frac{1}{2} V^2 + gz \right) \right]_2$$

Dividing by $|\dot{m}|$ and canceling $i_{\text{cst.}}$, as follows,

$$0 = - \left(i_{\text{cst.}} + \frac{p}{\rho_{\text{cst.}}} + \frac{1}{2} V^2 + gz \right)_1 + \left(i_{\text{cst.}} + \frac{p}{\rho_{\text{cst.}}} + \frac{1}{2} V^2 + gz \right)_2$$

$$0 = - \left(\frac{p}{\rho_{\text{cst.}}} + \frac{1}{2} V^2 + gz \right)_1 + \left(\frac{p}{\rho_{\text{cst.}}} + \frac{1}{2} V^2 + gz \right)_2$$

and multiplying by the (constant and uniform) density ρ , we obtain the *Bernoulli equation*, with all terms having dimensions of pressure:

$$\left(p + \frac{1}{2} \rho V^2 + \rho gz \right)_1 = \left(p + \frac{1}{2} \rho V^2 + \rho gz \right)_2 \quad (2/20)$$

This equation describes the properties of a fluid particle in a steady, incompressible, friction-less flow with no energy transfer.

2.6.2 Reality

Let us insist on the incredibly frustrating restrictions brought by the five conditions above:

1. Steady flow.
This constrains us to continuous flows with no transition effects, which is a reasonable limit;
2. Incompressible flow.
We cannot use this equation to describe flow in compressors, turbines, diffusers, nozzles, nor in flows where $M > 0,6$.
3. No heat or work transfer.
We cannot use this equation in a machine (e.g. in pumps, turbines, combustion chambers, coolers).
4. No friction.
This is a tragic restriction! We cannot use this equation to describe a turbulent or viscous flow, e.g. near a wall or in a wake.



Video: The Bernoulli equation will kill you

by Olivier Cleynen (CC-BY)
<https://youtu.be/4RBxVepjQc>

5. One-dimensional flow.

This equation is only valid if we know precisely the trajectory of the fluid whose properties are being calculated.

Of course, we can overcome those shortcomings by adding one extra (negative) term called “ Δp_{loss} ” to eq. 2/20, which lumps together all of the effects unaccounted for. In this way, we obtain the *Bernoulli equation with losses*:

$$\left(p + \frac{1}{2}\rho V^2 + \rho g z\right)_1 = \left(p + \frac{1}{2}\rho V^2 + \rho g z\right)_2 + \Delta p_{\text{loss}} \quad (2/21)$$

There are indeed cases where the pressure losses due to the imperfection of the flow are well-understood, and can be easily quantified. This is true of flow in pipes, for example (we study those in chapter 7). In those cases, eq. 2/21 is extremely useful.

Nevertheless, this approach is also easily misused. In a fluid flow where several of the restrictions above do not hold—and many such flows can be found in everyday life as well as engineering applications—equation 2/21 will betray its users. Convince yourself that *any wrong equation* can be made correct by adding an unknown “bucket” term at the end: for example $2 + 3 = -18 + \Delta p_{\text{loss}}$.

Advice from an expert

In case you are not sure whether the Bernoulli equation applies, *start from an energy balance equation*. Crossing out the terms that do not apply will force you to question their importance (e.g. is heat transfer really negligible? etc.). If you do not come to a conclusive end, do not remove terms that are inconvenient. The unfortunate reality is that in fluid mechanics, the energy balance equation contains many terms, with disproportionate values, and using it alone is not enough to solve most practical problems.



Advice from an expert

Among the five restrictions listed, the last is the most severe, and the most often forgotten: *the Bernoulli equation does not allow us to predict the trajectory of fluid particles*. Just like all of the other equations in this chapter, it requires a control volume with a known inlet and a known outlet. If you find yourself drawing out flow streamlines and interpreting the result with the Bernoulli equation, you are running astray. The tools you need to do this correctly are waiting for us in chapter 6 (*Prediction of fluid flows*).

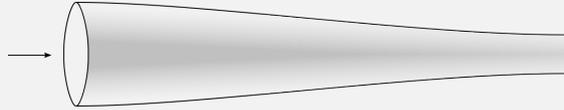


2.7 Solved problems

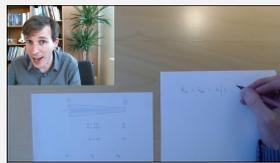
Flow in a nozzle

Water is flowing through a nozzle, where the diameter decreases gently from 2 m^2 to 1 m^2 . The flow is so smooth that energy dissipation due to wall friction is negligible.

The water enters the nozzle with a uniform velocity of 3 m s^{-1} .



What is the mass flow? What is the outlet velocity? And what is the pressure change across the pipe?



See this solution worked out step by step on YouTube
<https://youtu.be/ZqvZTQu8SgA> (CC-BY Olivier Cleynen)

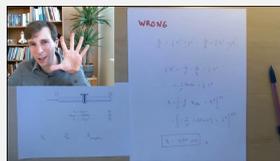
Flow through a valve

Water is flowing through a straight pipe with constant diameter. The mass flow entering the pipe is 2 kg s^{-1} , and it enters the pipe with a uniform velocity of 2 m s^{-1} .

In the middle of the pipe length, a valve is installed, which causes the pressure drop: $\Delta p_{\text{valve}} = -3,5\text{ kPa}$.



What is the outlet velocity? What is the net force on the fluid as it transits? What is the power dissipated as friction?



See this solution worked out step by step on YouTube
<https://youtu.be/OEfMpXtkCQM> (CC-BY Olivier Cleynen)

Problem sheet 2: Analysis of existing flows with one dimension

last edited April 15, 2021
by Olivier Cleynen – <https://fluidmech.ninja/>

Except otherwise indicated, assume that:

The atmosphere has $p_{\text{atm.}} = 1 \text{ bar}$; $\rho_{\text{atm.}} = 1,225 \text{ kg m}^{-3}$; $T_{\text{atm.}} = 11,3 \text{ }^\circ\text{C}$; $\mu_{\text{atm.}} = 1,5 \cdot 10^{-5} \text{ Pa s}$

Air behaves as a perfect gas: $R_{\text{air}}=287 \text{ J kg}^{-1} \text{ K}^{-1}$; $\gamma_{\text{air}}=1,4$; $c_{p \text{ air}}=1005 \text{ J kg}^{-1} \text{ K}^{-1}$; $c_{v \text{ air}}=718 \text{ J kg}^{-1} \text{ K}^{-1}$

Liquid water is incompressible: $\rho_{\text{water}} = 1000 \text{ kg m}^{-3}$, $c_{p \text{ water}} = 4180 \text{ J kg}^{-1} \text{ K}^{-1}$

Balance of mass in a fixed control volume with steady flow:

$$0 = \Sigma [\rho V_{\perp} A]_{\text{incoming}} + \Sigma [\rho V_{\perp} A]_{\text{outgoing}} \quad (2/3)$$

where V_{\perp} is negative inwards, positive outwards.

Balance of momentum in a fixed control volume with steady flow:

$$\vec{F}_{\text{net on fluid}} = \Sigma [\rho V_{\perp} A \vec{V}]_{\text{incoming}} + \Sigma [\rho V_{\perp} A \vec{V}]_{\text{outgoing}} \quad (2/7)$$

where V_{\perp} is negative inwards, positive outwards.

Balance of energy in a fixed control volume with steady flow:

$$\begin{aligned} \dot{Q}_{\text{net}} + \dot{W}_{\text{shaft, net}} = & \Sigma \left[\dot{m} \left(i + \frac{p}{\rho} + \frac{1}{2} V^2 + gz \right) \right]_{\text{in}} \\ & + \Sigma \left[\dot{m} \left(i + \frac{p}{\rho} + \frac{1}{2} V^2 + gz \right) \right]_{\text{out}} \end{aligned} \quad (2/18)$$

where \dot{m} is negative inwards, positive outwards.

2.1 Reading quiz

Once you are done with reading the content of this chapter, you can go take the associated quiz at <https://elearning.ovgu.de/course/view.php?id=7199>

In the winter semester, quizzes are not graded.



2.2 Pipe expansion without losses

Water flows from left to right in a pipe, as shown in fig. 2.3. On the left, the diameter is 8 cm, the water arrives with a uniform velocity of $1,5 \text{ m s}^{-1}$. The diameter increases gently until it reaches 16 cm; the expansion is smooth, so that losses (specifically, energy losses due to wall friction and flow separation) are negligible.

2.2.1. What are the mass and volume flows at inlet and outlet?

2.2.2. What is the average velocity of the water at the right end of the expansion?

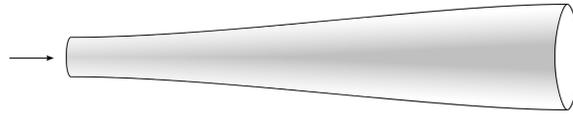


Figure 2.3: A simple pipe expansion, with water flowing from left to right.

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2.2.3. What is the pressure change in the water across the expansion?

The volume flow of water in the pipe is now doubled.

2.2.4. What is the new pressure change?

The water is drained from the pipe, and instead, air with density $1,225 \text{ kg m}^{-3}$ is flowed in the pipe, incoming with a uniform velocity of $1,5 \text{ m s}^{-1}$.

2.2.5. What is the new pressure change?

2.3 Pipe flow with losses

Water flows in a long pipe which has constant diameter; a valve is installed in the middle of the pipe length (fig. 2.4). Water arrives the pipe with a uniform velocity of $1,5 \text{ m s}^{-1}$ and the pipe diameter is 250 mm.

The pipe itself and the valve, together, induce a pressure loss which can be quantified using the dimensionless *loss coefficient* K_{valve} (we later will later encounter it as eq. 7/6 p. 137). With this tool, the pressure loss Δp_{valve} is related to the mean incoming speed V_{incoming} as:

$$K_{\text{valve}} \equiv \frac{|\Delta p_{\text{valve}}|}{\frac{1}{2}\rho V_{\text{incoming}}^2} = 2,6 \quad (2/22)$$

2.3.1. What is the outlet velocity of the water?

(note: this is a classical “trick” question! :-)

2.3.2. What is the drop in water pressure across the valve?

2.3.3. What is the power dissipated into internal energy?

2.3.4. If the heat losses of the pipe and valve are negligible, what is the temperature increase of the water?

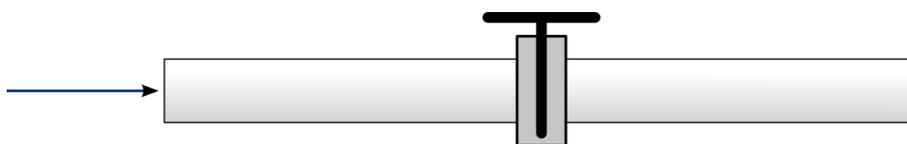


Figure 2.4: A simple, straight pipe, featuring a partially-open valve in the center

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2.4 Combustor from a jet engine

A jet engine is equipped with several *combustors* (sometimes also called *combustion chambers*). We are interested in fluid flow through one such combustor, shown in fig. 2.5. Air from the compressor enters the combustor, is mixed with fuel, and combustion occurs, which greatly increases the temperature and specific volume of the mix, before it is run through the turbine.

The conditions at inlet are as follows:

- Air mass flow: $0,5 \text{ kg s}^{-1}$;
- Air properties: 25 bar, $1\,050 \text{ }^\circ\text{C}$, 12 m s^{-1}
- Fuel mass flow: 5 g s^{-1} .

At the outlet, the hot gases have pressure 24,5 bar and temperature $1\,550 \text{ }^\circ\text{C}$, and exit with a speed of 50 m s^{-1} .

We assume that the incoming air and outgoing gas have the same thermodynamic properties: $c_v = 718 \text{ J kg}^{-1} \text{ K}^{-1}$, $R_{\text{air}} = 287 \text{ J kg}^{-1} \text{ K}^{-1}$.

2.4.1. What are the mass flows at inlet and outlet?

2.4.2. What are the volume flows at inlet and at outlet?

2.4.3. What is the power provided to the flow as heat?

2.4.4. What is the net force exerted on the gas as it travels through the combustor?

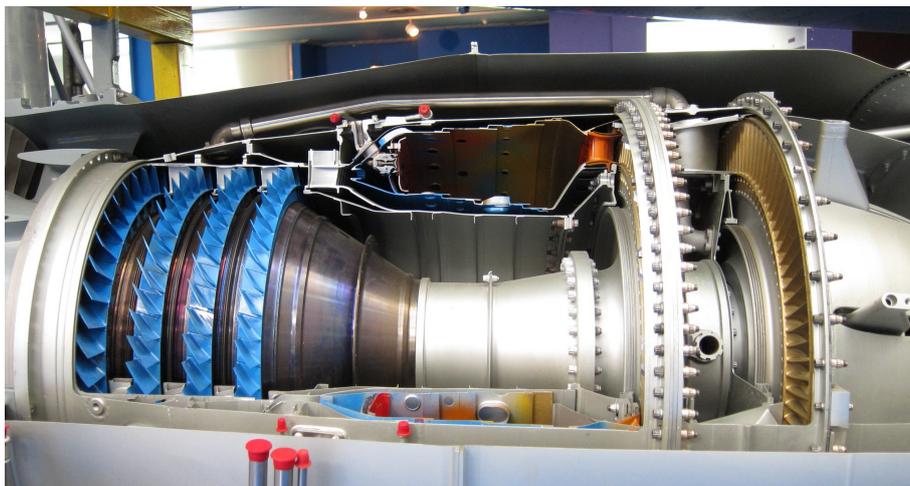


Figure 2.5: A combustor in a sectioned jet engine (here, a Turboméca Adour). Air enters from the left, out of the compressor (whose blades are painted blue). It leaves the combustor on the right side, into the turbine. In the combustor, high-temperature, steady combustion takes place.

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2.5 Water jet on a truck

A water nozzle shoots water towards the back of a small stationary van. It has a 3 cm^2 cross-sectional area, and the water speed at the nozzle outlet is $V_{\text{jet}} = 20 \text{ m s}^{-1}$. As the horizontal water jet hits the back of the van, it is split in two symmetrical vertical flows (fig. 2.6). The two opposite vertical jets have same mass flow and same velocity ($V_2 = V_3 = 20 \text{ m s}^{-1}$).

2.5.1. What is the net force exerted on the water by the truck?

2.5.2. What is the net force exerted on the truck by the water?

Now, the truck moves longitudinally in the same direction as the water jet, with a speed $V_{\text{truck}} = 15 \text{ m s}^{-1}$.

(This is a crude conceptual setup, which allows us to approach conceptually the case where water acts on the blades of a turbine.)

2.5.3. What is the new force exerted by the water on the truck?

2.5.4. What is the mechanical power transmitted to the truck?

2.5.5. How would the power be modified if the volume flow was kept constant, but the diameter of the nozzle was reduced? (briefly justify your answer, e.g. in 30 words or less)

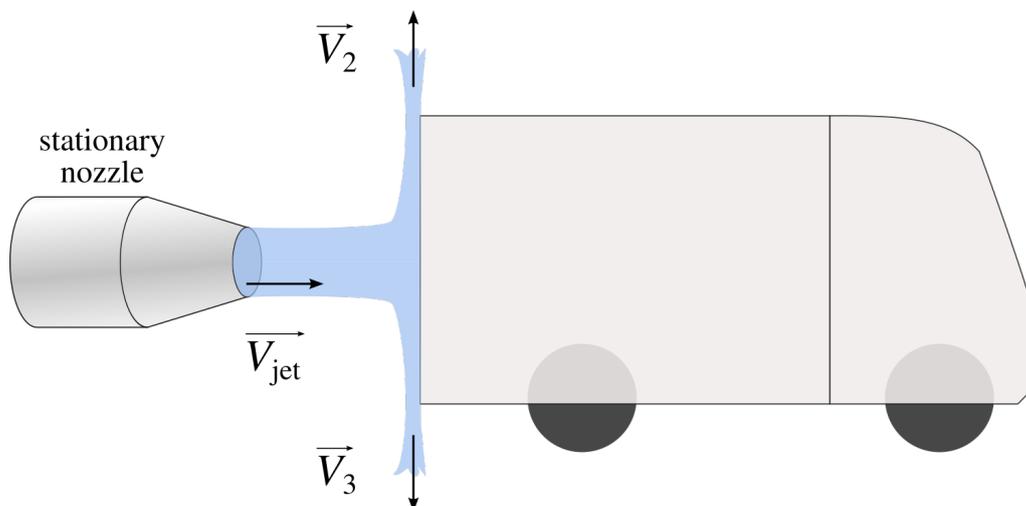


Figure 2.6: A water jet flowing out of a nozzle (left), and impacting the vertical back surface of a small electric truck, on the right.

Figure CC-0 Olivier Cleynen

2.6 High-speed gas flow

Scientists build a very high-speed wind tunnel. For this, they build a large compressed air tank. Air escapes from the tank into a pipe which decreasing cross-section, as shown in fig. 2.7. The pipe diameter reaches a minimum (at the tunnel *throat*), and then it expands again, before discharging into the atmosphere.

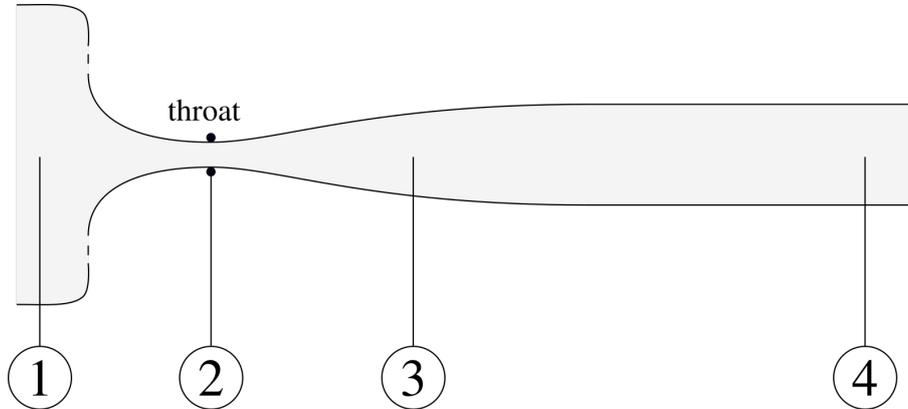


Figure 2.7: A converging-diverging nozzle. Air flows from the left tank to the right outlet, with a contraction in the middle.

Figure CC-0 Olivier Cleynen

For simplicity, we assume that heat losses through the tunnel walls are negligible, and that the fluid has uniformly-distributed velocity in cross-sections of the pipe.

In the tank (point 1), the air is stationary, with pressure 7,8 bar and temperature 246,6 °C. At the throat (point 2), the pressure and temperature have dropped to 4,2 bar and 160 °C. The velocity has reached 417,2 m s⁻¹. The throat cross-section is 0,01 m².

2.6.1. What is the mass flow through the tunnel?

2.6.2. What is the kinetic energy per unit mass of the air at the throat?

Downstream of the throat, the pressure keeps dropping. By the time it reaches a point 3, the air has seen its pressure and temperature drop to 1,38 bar and 43 °C.

2.6.3. What is the fluid velocity at point 3?

(if you need to convince yourself that $A_3 > A_1$, you may also calculate the cross-section area)

2.6.4. What is the net force exerted on the fluid between the points 2 and 3?

2.6.5. What is the kinetic energy per unit mass of the air at point 3?

Once it has passed point 3, the air undergoes complex loss-inducing evolutions (including going through a *shock wave*, where its properties change very suddenly), before it discharges into the atmosphere (point 4) with pressure 1 bar and temperature 165 °C.

2.6.6. What is the fluid velocity at outlet?

2.6.7. What is the outlet cross-section area?

2.6.8. What is the net force exerted on the fluid between section 3 and the outlet?

2.6.9. What is the kinetic energy per unit mass of the air at the outlet?

Answers

2.2 p. 45

2.2.1 At both inlet and outlet, $\dot{m} = 7,53 \text{ kg s}^{-1}$ and $\dot{\mathcal{V}} = 7,53 \text{ L s}^{-1}$

2.2.2 $V_2 = 0,375 \text{ m s}^{-1}$

2.2.3 $\Delta p_{1 \rightarrow 2} = +1\,054 \text{ Pa}$

2.2.4 $\Delta p_{3 \rightarrow 4} = +4\,218 \text{ Pa}$

2.2.5 $\Delta p_{5 \rightarrow 6} = +1,29 \text{ Pa}$

2.3 p. 46

2.3.1 $V_2 = V_1 = 1,5 \text{ m s}^{-1}$ by application of the mass balance equation; although a mis-application of the energy equation would suggest otherwise

2.3.2 With eq. 2/22, $\Delta p_{\text{valve}} = -2\,925 \text{ Pa}$

2.3.3 $\dot{m}\Delta i = +215,37 \text{ W}$

2.3.4 With eq. 2/18, $\Delta T = +0,7 \text{ mK}$ (very small!)

2.4 p. 47

2.4.1 $|\dot{m}_1| = 0,005 + 0,5 \text{ kg s}^{-1}$ and $|\dot{m}_2| = 0,505 \text{ kg s}^{-1}$

2.4.2 $\dot{\mathcal{V}}_1 = 0,0759 \text{ m}^3 \text{ s}^{-1}$ & $\dot{\mathcal{V}}_2 = 0,1078 \text{ m}^3 \text{ s}^{-1}$ (there is no volume balance equation!)

2.4.3 $\dot{Q} = +261 \text{ kW}$ (using $V_2 = 50 \text{ m s}^{-1}$)

2.4.4 $F_{\text{net}} = +19,25 \text{ N}$ (in flow-wise direction)

2.5 p. 48

2.5.1 $F_{\text{net on water}} = -120 \text{ N}$

2.5.2 $\vec{F}_{\text{water/truck}} = -\vec{F}_{\text{net on water}}$

2.5.3 $F_{\text{net on water}} = -7,5 \text{ N}$

2.5.4 $\dot{W}_{\text{truck}} = 112,5 \text{ W}$

2.6 p. 49

2.6.1 With eq. 2/18, $V_2 = 417,2 \text{ m s}^{-1}$, and so $\dot{m}_2 = \dot{m} = 14,1 \text{ kg s}^{-1}$

2.6.2 $e_{k2} = 87,03 \text{ kJ kg}^{-1}$

2.6.3 With eq. 2/18, $V_3 = 638,71 \text{ m s}^{-1}$ (you may then calculate ρ_3 and obtain $A_3 > A_2$ even though $V_3 > V_2$, a classical feature of supersonic flows)

2.6.4 $F_{\text{net}2 \rightarrow 3} = +3,137 \text{ kN}$

2.6.5 $e_{k3} = 204,61 \text{ kJ kg}^{-1}$

2.6.6 $V_4 = 405 \text{ m s}^{-1}$

2.6.7 $\rho_4 = 0,7952 \text{ kg s}^{-1}$ and so $A_4 = 0,0438 \text{ m}^2$

2.6.8 $F_{\text{net}3 \rightarrow 4} = -3,309 \text{ kN}$ (so, against the flow direction)

2.6.9 $e_{k4} = 82,01 \text{ kJ kg}^{-1}$