Fluid Mechanics
Chapter 2 – Effects of shear

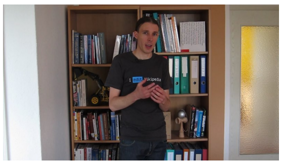
last edited April 18, 2018

These lecture notes are based on textbooks by White [13], Çengel & al.[16], and Munson & al.[18].

2.1 Motivation

In fluid mechanics, only three types of forces apply to fluid particles: forces due to gravity, pressure, and shear. This chapter focuses on shear, and should allow us to answer two questions:

- How is the effect of shear described and quantified?
- What are the shear forces generated on walls by simple flows?



Video: pre-lecture briefing for this chapter

by o.c. (CC-BY)
https://youtu.be/BKKXJWgLwJg

2.2 Concept of shear

We approached the concept of shear in the introductory chapter with the notion that it represented force parallel to a given flat surface (eq. 0/16), for example a flat plate of area A:

$$\tau \equiv \frac{F_{\parallel}}{A} \tag{2/1}$$

Like we did with pressure, to appreciate the concept of shear in fluid mechanics, we need to go beyond this equation.

2.2.1 The direction of shear

Already from the definition in eq. 2/1 we can appreciate that "parallel to a flat plate" can mean a multitude of different directions, and so that we need more than one dimension to represent shear. Furthermore, much in the same way as we did for pressure, we do away with the flat plate and accept that shear is a *field*, i.e. it is an effort applying not only upon solid objects but also upon and within fluids themselves. We replace eq. 2/1 with a more general definition:

$$\vec{\tau} \equiv \lim_{A \to 0} \frac{\vec{F}_{\parallel}}{A} \tag{2/2}$$

Video: cloud movements in a time-lapse video on an interesting day are evidence of a highly-strained atmosphere: pilots and meteorologists refer to this as *wind shear*.

by Y:StormsFishingNMore (STYL)
https://youtu.be/LjWeYPEmCk8

Contrary to pressure, shear is not a scalar, i.e. it can (and often does) take different values in different directions. At a given *point* in space we represent it as a vector $\vec{\tau} = \left(\tau_x, \tau_y, \tau_y\right)$, and in a fluid, there is a shear *vector field*:

$$\vec{\tau}_{(x,y,z,t)} \equiv \begin{pmatrix} \tau_x \\ \tau_y \\ \tau_z \end{pmatrix}_{(x,y,z,t)} \tag{2/3}$$

2.2.2 Shear on an infinitesimal volume

Describing the changes in space of the shear vector field requires another mathematical dimension (called *order*).
Instead of a flat plate, let us consider an infinitesimally small cube within the fluid (fig. 2.1). Because the cube is immersed inside a vector field, the shear vector exerting on each of its six faces may be different.

In order to express the efforts on any given face, we express a component of shear with two subscripts, the first indicating the direction normal to the surface of interest, and the second indicating the direction of the effort. For example, $\vec{\tau}_{xy}$ represents the shear in the y-direction on a surface perpendicular to the x-direction. On this face, the shear vector would be:

$$\vec{\tau}_{xj} = \vec{\tau}_{xx} + \vec{\tau}_{xy} + \vec{\tau}_{xz} \tag{2/4}$$
$$= \tau_{xx}\vec{i} + \tau_{xy}\vec{j} + \tau_{xz}\vec{k} \tag{2/5}$$

where the subscript xj indicates all of the directions ($j = x, y, z$) on a face perpendicular to the x-direction.

In eq. 2/4, the reader may be surprised to see the term τ_{xx} appear — a shear effort perpendicular to the surface of interest. This is because the faces of the infinitesimal cube studied here (shown in fig. 2.1) are not solid. They are permeable, and the local velocity on each one may (in fact, must, if there is to be any flow) include a component of velocity through the face of the cube. Thus, there is no reason for the shear effort, which is three-dimensional, to be aligned along each flat surface. As the fluid travels across any face, it can be sheared (which results in strain) in any arbitrary direction, regardless of the local pressure — and thus shear can and most often does have a component (τ_{ii}) perpendicular to an arbitrary surface inside a fluid.

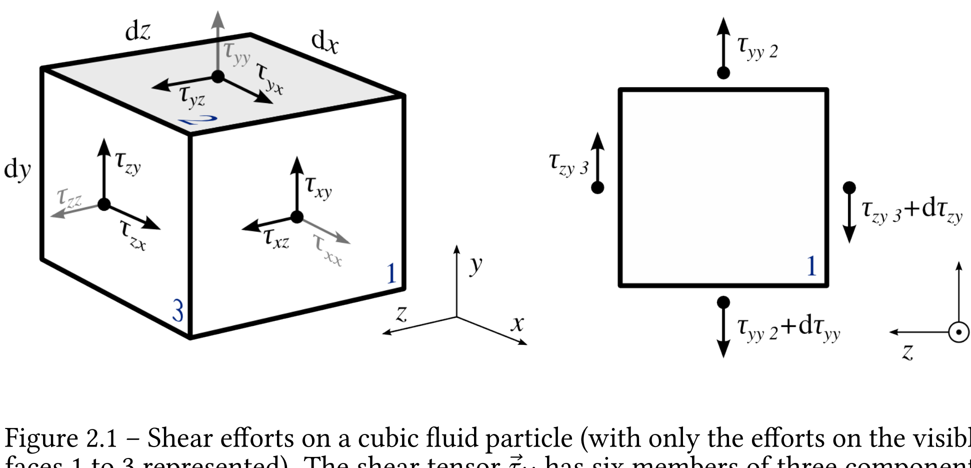


Figure 2.1 – Shear efforts on a cubic fluid particle (with only the efforts on the visible faces 1 to 3 represented). The shear tensor $\vec{\tau}_{ij}$ has six members of three components each.

Figure CC-0 o.c.

Now, the net shear effect on the cube will have *eighteen* components: one tree-dimensional vector for each of the six faces. Each of those components may take a different value. The net shear could perhaps be represented as en entity —a *tensor*— containing six vectors $\vec{\tau}_1, \vec{\tau}_2, \vec{\tau}_3 \ldots \vec{\tau}_6$. By convention, however, shear is notated using only three vector components: one for each pair of faces. Shear efforts on a volume are thus represented with a *tensor field* $\vec{\tau}_{ij}$:

$$\vec{\tau}_{ij} \equiv \begin{pmatrix} \vec{\tau}_{xj} \\ \vec{\tau}_{yj} \\ \vec{\tau}_{zj} \end{pmatrix} \equiv \begin{pmatrix} \vec{\tau}_{xj\ \{1,4\}} \\ \vec{\tau}_{yj\ \{2,5\}} \\ \vec{\tau}_{zj\ \{3,6\}} \end{pmatrix}$$

$$\vec{\tau}_{ij} \equiv \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \tag{2/6}$$

In this last equation 2/6, each of the nine components of the tensor acts as the container for two contributions: one for each of the two faces perpendicular to the direction expressed in its first subscript.

So much for the shear *effort* on an element of fluid. What about the net *force* due to shear on the fluid element? Not every element counts: part of the shear will accelerate (change the velocity vector) the particle, while part of it will merely strain (deform) the particle. Quantifying this force thus requires making a careful selection within the eighteen components of $\vec{\tau}_{ij}$. We may start with the x-direction, which consists of the sum of the component of shear in the x-direction on each of the six cube faces:

$$\begin{aligned} \vec{F}_{\text{shear } x} = {} & S_3 \vec{\tau}_{zx\ 3} - S_6 \vec{\tau}_{zx\ 6} \\ & + S_2 \vec{\tau}_{yx\ 2} - S_5 \vec{\tau}_{yx\ 5} \\ & + S_1 \vec{\tau}_{xx\ 1} - S_4 \vec{\tau}_{xx\ 4} \end{aligned} \tag{2/7}$$

Given that $S_3 = S_6 = \mathrm{d}x\,\mathrm{d}y$, that $S_2 = S_5 = \mathrm{d}x\,\mathrm{d}z$ and that $S_1 = S_4 = \mathrm{d}z\,\mathrm{d}y$, this is re-written as:

$$\begin{aligned} \vec{F}_{\text{shear } x} = {} & \mathrm{d}x\,\mathrm{d}y (\vec{\tau}_{zx\ 3} - \vec{\tau}_{zx\ 6}) \\ & + \mathrm{d}x\,\mathrm{d}z (\vec{\tau}_{yx\ 2} - \vec{\tau}_{yx\ 5}) \\ & + \mathrm{d}z\,\mathrm{d}y (\vec{\tau}_{xx\ 1} - \vec{\tau}_{xx\ 4}) \end{aligned} \tag{2/8}$$

In the same way we did with pressure in chapter 1 (§1.3.3 p.31), we express each pair of values as derivative with respect to space multiplied by an infinitesimal distance:

$$\begin{aligned} \vec{F}_{\text{shear } x} &= \mathrm{d}x\,\mathrm{d}y \left(\mathrm{d}z \frac{\partial \vec{\tau}_{zx}}{\partial z} \right) + \mathrm{d}x\,\mathrm{d}z \left(\mathrm{d}y \frac{\partial \vec{\tau}_{yx}}{\partial y} \right) + \mathrm{d}z\,\mathrm{d}y \left(\mathrm{d}x \frac{\partial \vec{\tau}_{xx}}{\partial x} \right) \\ &= \mathrm{d}\mathcal{V} \left(\frac{\partial \vec{\tau}_{zx}}{\partial z} + \frac{\partial \vec{\tau}_{yx}}{\partial y} + \frac{\partial \vec{\tau}_{xx}}{\partial x} \right) \end{aligned} \tag{2/9}$$

If we make use of the operator *divergent* (see also Appendix A2 p.217), written $\vec{\nabla}\cdot$:

$$\vec{\nabla}\cdot \equiv \frac{\partial}{\partial x}\vec{i}\cdot + \frac{\partial}{\partial y}\vec{j}\cdot + \frac{\partial}{\partial z}\vec{k}\cdot \tag{2/10}$$

$$\vec{\nabla}\cdot\vec{A} \equiv \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{2/11}$$

$$\vec{\nabla}\cdot\vec{A}_{ij} \equiv \begin{pmatrix} \frac{\partial A_{xx}}{\partial x} + \frac{\partial A_{yx}}{\partial y} + \frac{\partial A_{zx}}{\partial z} \\ \frac{\partial A_{xy}}{\partial x} + \frac{\partial A_{yy}}{\partial y} + \frac{\partial A_{zy}}{\partial z} \\ \frac{\partial A_{xz}}{\partial x} + \frac{\partial A_{yz}}{\partial y} + \frac{\partial A_{zz}}{\partial z} \end{pmatrix} = \begin{pmatrix} \vec{\nabla}\cdot\vec{A}_{ix} \\ \vec{\nabla}\cdot\vec{A}_{iy} \\ \vec{\nabla}\cdot\vec{A}_{iz} \end{pmatrix} \tag{2/12}$$

we can re-write eq. 2/9 and see that the net shear force in the x-direction is equal to the particle volume times the divergent of the shear in the x-direction:

$$\vec{F}_{\text{shear } x} = \mathrm{d}\mathcal{V}\, \vec{\nabla}\cdot\vec{\tau}_{ix} \tag{2/13}$$

The y- and z-direction are taken care of in the same fashion, so that we can gather up our puzzle pieces and express *the force per volume due to shear as the divergent of the shear tensor*:

$$\vec{F}_{\text{shear}} = \begin{pmatrix} F_{\text{shear } x} \\ F_{\text{shear } y} \\ F_{\text{shear } z} \end{pmatrix} = \mathrm{d}\mathcal{V} \begin{pmatrix} |\vec{\nabla}\cdot\vec{\tau}_{ix}| \\ |\vec{\nabla}\cdot\vec{\tau}_{iy}| \\ |\vec{\nabla}\cdot\vec{\tau}_{iz}| \end{pmatrix} = \mathrm{d}\mathcal{V}\, \vec{\nabla}\cdot\vec{\tau}_{ij} \tag{2/14}$$

This exploration goes beyond the theory required to to through this chapter (we will come back to it when we start concerning ourselves with the dynamics of fluid particles in chapter 4, where the divergent of shear will make part of the glorious *Cauchy equation*). It suffices for now to sum up our findings as follows:

- Shear at a point in space has three components — it is a vector field;
- The effect of shear on a volume of fluid has eighteen components – it is a second-order tensor field;
- The net force due to shear on a volume of fluid, expressed using the divergent of the shear tensor, has three components — it is a vector field.

2.3 Slip and viscosity

2.3.1 The no-slip condition

We observe that whenever we measure the velocity of a fluid flow along a solid wall, the speed tends to zero as we approach the wall surface. In other words, the fluid "*sticks*" to the surface regardless of the overall faraway flow velocity. This phenomenon is called the *no-slip condition* and is of paramount importance in fluid mechanics. One consequence of this is that fluid flows near walls are dominated by viscous effects (internal friction) due to the large velocity gradients.

2.3.2 Viscosity

When a solid wall is moved longitudinally within a fluid, the fluid generates an opposing friction force through viscous effects (fig. 2.2). We call *viscosity* μ the ratio between the fluid velocity gradient and the shear effort. For example, the norm of the shear $\vec{\tau}_{xy}$ on a surface perpendicular to the x-direction, in the y-direction, can be expressed as:

$$||\vec{\tau}_{xy}|| = \mu \frac{\partial u_y}{\partial x} = \mu \frac{\partial v}{\partial x}$$

In the general case, viscosity is defined as the (scalar) ratio between the norm of shear and the corresponding strain rate. The strain rate corresponding to the shear in the j-direction is the rate of change in the i-direction of the velocity in the j-direction ($\partial u_j / \partial i$):

$$\mu \equiv \frac{||\vec{\tau}_{ij}||}{\left(\frac{\partial u_j}{\partial i}\right)} \tag{2/15}$$

$$||\vec{\tau}_{ij}|| = \mu \frac{\partial u_j}{\partial i} \tag{2/16}$$

in which the subscript i is an arbitrary direction (x, y or z) and j is the direction following it in order (e.g. $j = z$ when $i = y$);
and where μ is the viscosity (or *dynamic viscosity*) (Pa s).

Viscosity μ is measured in Pa s, which is the same as $\mathrm{N\,s\,m^{-2}}$ or $\mathrm{kg\,m^{-1}\,s^{-1}}$. It has historically been measured in poise (1 poise ≡ 0,1 Pa s).

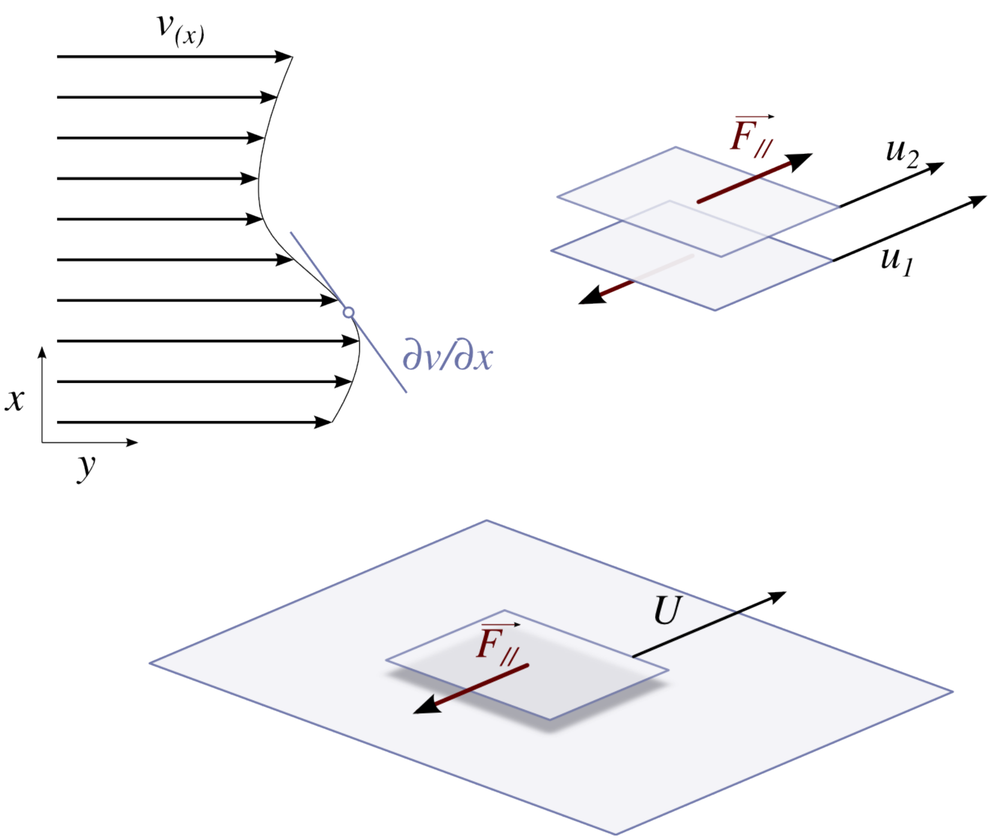


Figure 2.2 – Any velocity gradient $\frac{\partial u_y}{\partial x}$ in the flow results in a shear force $F_{||}$ in the direction y. The ratio between the shear and the velocity gradient is called *viscosity*.

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The concept of *kinematic viscosity* ν (Greek letter *nu*) is sometimes used instead of the dynamic viscosity; it is defined as

$$\nu \equiv \frac{\mu}{\rho} \tag{2/17}$$

where ν is measured in $\mathrm{m^2\,s^{-1}}$.

2.3.3 Newtonian Fluid

Fluids for which μ is independent from $\frac{\partial u_j}{\partial i}$ are called *Newtonian fluids*.

Most fluids of interest in engineering fluid mechanics (air, water, exhaust gases, pure gases) can be safely modeled as Newtonian fluids. Their viscosity μ varies slightly with pressure and mildly with temperature – in our study of fluid mechanics, we will not take these dependencies into account.
The values of viscosity vary very strongly from one fluid to another: for example, honey is roughly ten thousand times more viscous than water, which is roughly a hundred times more viscous than ambient air. The viscosities of various fluids are quantified in fig. 5.11 p.119.

Oil-based paint, blood and jelly-based fluids are strongly non-Newtonian; they require more complex viscosity models (fig. 2.3).

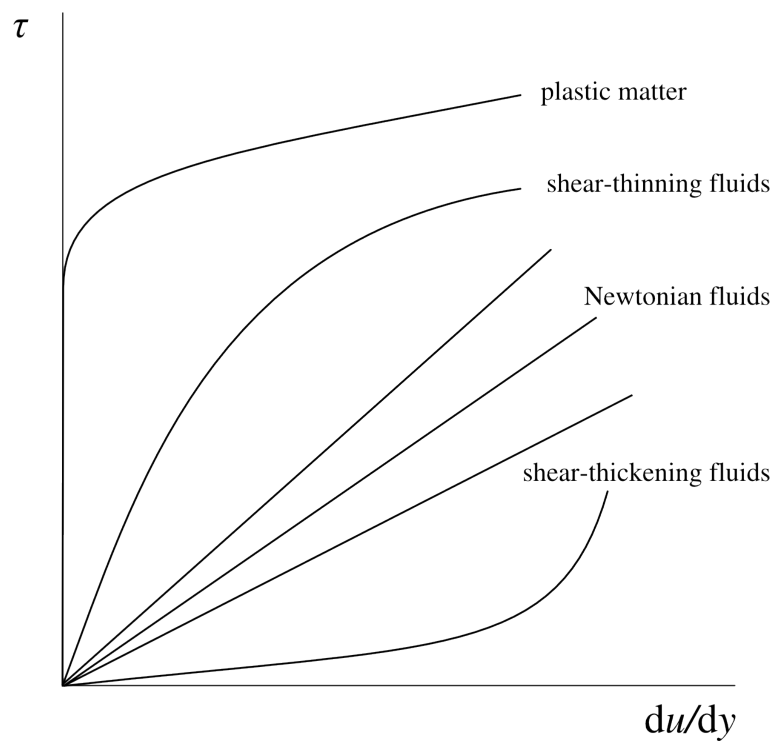

Figure 2.3 – Various possible viscosity characteristics of fluids. Those for which μ is independent of $\partial u_j/\partial i$ are called *Newtonian*.

2.4 Wall shear forces

The calculation of forces due to shear is very similar to that which we used in the previous chapter with pressure.

When the shear $\vec{\tau}$ exerted on a flat surface of area S is uniform, the resulting force F is easily easily calculated ($F = \tau_{\text{cst.}} S$). In the more general case of a three-dimensional object immersed in a fluid with non-uniform shear, the force must be expressed as a vector and obtained by integration:

$$\vec{F}_{\text{shear}} = \int_S \mathrm{d}\vec{F} = \int_S \vec{\tau}_{\text{surface}} \, \mathrm{d}S \tag{2/18}$$

Much like equation 1/13 in the previous chapter, eq. 2/18 is easily implemented in software algorithms to obtain numerically, for example, the force resulting from shear due to fluid flow around a body such as an aircraft wing. In our academic study of fluid mechanics, however, we will restrict ourselves to the simpler cases where the surface is perfectly flat and the shear has uniform direction. The force in eq. 2/18, exterting in the i-direction due to shear on a surface perpendicular to the j-direction, then becomes:

$$F_{\text{shear } ji} = \int_S \mathrm{d}F = \int_S ||\vec{\tau}_{ji}|| \, \mathrm{d}S \tag{2/19}$$

What is required to calculate the scalar F in eq. 2/19 is an expression of τ as a function of S. We proceed like we did in the previous chapter, splitting $\mathrm{d}S$ as $\mathrm{d}S = \mathrm{d}i \, \mathrm{d}k$ before proceeding with the calculation starting from;

$$F_{\text{shear}ji} = \iint \tau_{ji \ (i,k)} \, \mathrm{d}i \, \mathrm{d}k \tag{2/20}$$

$$= \mu \iint \frac{\partial u_i}{\partial j} \, \mathrm{d}i \, \mathrm{d}k \tag{2/21}$$

where i is the direction of the force;
j is the direction perpendicular to the flat surface;
and k is the third (orthonormal) direction.

The above expression 2/21 is perhaps easier to read when it is developed. For example, the shear force $\vec{F}_{\text{shear}yi}$ exerting on a plate perpendicular to the y-direction is:

$$\vec{F}_{\text{shear}yi} = \begin{pmatrix} F_{\text{shear}yx} \\ 0 \\ F_{\text{shear}yz} \end{pmatrix}$$

with

$$F_{\text{shear}yx} = \mu \iint \frac{\partial u}{\partial y} \, \mathrm{d}x \, \mathrm{d}z$$

$$F_{\text{shear}yz} = \mu \iint \frac{\partial w}{\partial y} \, \mathrm{d}x \, \mathrm{d}z$$

Just like for pressure, the moment $\vec{M}_{\text{X}j}$ generated about a point X by the shear forces in the i-direction on a plane surface perpendicular to the j-direction can be calculated as

$$\vec{M}_{\text{X}j} = \int_S \mathrm{d}\vec{M}_{\text{X}} = \int_S \vec{r}_{\text{X}F} \wedge \mathrm{d}\vec{F} = \int_S \vec{r}_{\text{X}F} \wedge \vec{\tau}_{ij} \, \mathrm{d}S \tag{2/22}$$

If the point X is in the plane of the surface of interest, this can be expressed as:

$$M_{\mathrm{X}j} = \mu \iint r_{\mathrm{X}F} \frac{\partial u_i}{\partial j} \, \mathrm{d}i \, \mathrm{d}k \tag{2/23}$$

$$M_{\mathrm{X}y} = \mu \iint r_{\mathrm{X}F} \frac{\partial u_x}{\partial y} \, \mathrm{d}x \, \mathrm{d}z \tag{2/24}$$