## Appendix

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A1  Notation

By definition. The \( = \) symbol sets the definition of the term on its left (which does not depend on previous equations).

(\text{dot above symbol}) Time rate: \( \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} \). For example, \( \dot{Q} \) is the rate of heat (in watts) representing a heat quantity \( Q \) (in joules) every second.

(\text{bar above symbol}) Time-average: \( \overline{\mathbf{A}} = \text{avg}(A) = \text{avg}(\overline{A} + A') \). The prime symbol indicates the instantaneous fluctuation around the average.

\( \Delta \) Indicates a net difference between two values: \( (\Delta X)_{B \rightarrow A} = X_B - X_A \). Can be negative.

\textit{italics} Physical properties (e.g. mass \( m \), temperature \( T \)).

\textbf{straight subscripts} Points in space or in time (temperature \( T_A \) at point A).
Subscripts “cst” indicate a constant property, “in” indicates “incoming” and “out” is “outgoing”.
Subscript “av.” indicates “average”.

\textbf{operators} Differential \( d \), partial differential \( \partial \), finite differential \( \delta \), total (alt.: substantial) derivative \( D/Dt \) (def. eq. 6/7 p. 117), exponential \( \exp x = e^x \), natural logarithm \( \ln x = \log_e x \).

\textbf{vectors} Vectors are written with an arrow. Velocity is \( \mathbf{V} = (u, v, w) \), alternatively written \( u_i = (u, v, w) \). The norm of a vector \( \mathbf{A} \) (positive or negative) is \( |\mathbf{A}| \), its length (always positive) is \( ||\mathbf{A}|| \).

\textbf{vector calculus}
Dot product \( \mathbf{A} \cdot \mathbf{B} \) (see §A2.1 p. 249);
Cross product \( \mathbf{A} \times \mathbf{B} \) (see §A2.2 p. 250);
Gradient \( \nabla A \) (def. eq. 4/11 p. 79, see also §A3.1 p. 252);
Divergent \( \nabla \cdot \mathbf{A} \) (def. eq. 5/14 p. 97, see also §A3.2 p. 252);
Laplacian \( \nabla^2 \mathbf{A} \) (def. eq. 6/38 p. 126, see also §A3.4 p. 253);
Curl \( \nabla \times \mathbf{A} \) (def. eq. A/32 p. 254, see also §A3.5 p. 254).

\textbf{units} Units are typed in roman (normal) font and colored gray (1 kg). In sentences units are fully-spelled and conjugated (one hundred watts). The liter is noted \( \text{L} \) to increase readability (1 L = \( 10^{-3} \) m\(^3\)). Units in equations are those from \textit{système international} (si) unless otherwise indicated.

\textbf{numbers} The decimal separator is a comma, the decimal exponent is preceded by a dot, integers are written in groups of three (1,234 \( \cdot \) \( 10^3 \) = 1234). Numbers are rounded up as late as possible and never in series. Leading and trailing zeros are never indicated.
A2 Vector operations

For a step-by-step revision of those notions and many more, written in a progressive, nonjudgmental way, with plenty of worked-out exercises, you can try John Bird’s Higher Engineering Mathematics [16].

A2.1 Vector dot product

The dot product of two vectors is a number defined as:

\[
\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta
\]  

(A/1)

where \( \theta \) is the angle separating the two vectors \( \vec{a} \) and \( \vec{b} \).

In this document, the dot product is always written with a median dot (\( \vec{a} \cdot \vec{b} \)), but in other literature, it is sometimes written with the \( \times \) symbol. Take care not to confuse it with the vector cross product (see §A2.2 p. 250).

It can be shown that the dot product of two vectors \( \vec{a}\{x_a, y_a, z_a\} \) and \( \vec{b}\{x_b, y_b, z_b\} \) can be quantified as:

\[
\vec{a} \cdot \vec{b} = x_a x_b + y_a y_b + z_a z_b
\]  

(A/2)

The dot product of two vectors is the same regardless of the order in which they are multiplied:

\[
\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}
\]  

(A/3)

It is easily shown using eq. (A/1) that:

\[
(\vec{a} \cdot -\vec{b}) = -(\vec{a} \cdot \vec{b})
\]  

(A/4)

Figure A.1 – Two vectors \( \vec{a} \) and \( \vec{b} \).
A2.2 Vector cross product

The cross product of two vectors is a vector written as:

\[ \mathbf{a} \times \mathbf{b} = \mathbf{c} \]  \hspace{1cm} (A/5)

The vector \( \mathbf{c} \) is so that:

- **its length** is equal to
  \[ c = a \cdot b \cdot \sin \theta \]  \hspace{1cm} (A/6)

- **its direction** is perpendicular to \( \mathbf{a} \) and \( \mathbf{b} \);

- **its orientation** is so that if \( \mathbf{b} \) is positioned at the end of \( \mathbf{a} \), then \( \mathbf{c} \) points away from a point from which the rotation generated \( \mathbf{b} \) is in the clockwise direction.

Describing \( \mathbf{c} \) requires a third dimension, even if \( \mathbf{a} \) et \( \mathbf{b} \) have only two dimensions.

In this document, the cross product is written with a wedge symbol (\( \mathbf{a} \wedge \mathbf{b} \)) but in the literature, it is often written with the symbol \( \times \). Make sure you do not confuse it with the dot product (§A2.2 p. 250).

It can be shown that the cross product \( \mathbf{c} \) of two vectors \( \mathbf{a}\{x_a, y_a, z_a\} \) et \( \mathbf{b}\{x_b, y_b, z_b\} \) is:

\[ \mathbf{c} = \begin{vmatrix} i & j & k \\ x_a & y_a & z_a \\ x_b & y_b & z_b \end{vmatrix} \] \hspace{1cm} (A/7)

So that one obtains:

\[ \mathbf{c} = \begin{vmatrix} y_a & z_a \\ y_b & z_b \end{vmatrix} \mathbf{i} - \begin{vmatrix} x_a & z_a \\ x_b & z_b \end{vmatrix} \mathbf{j} + \begin{vmatrix} x_a & y_a \\ x_b & y_b \end{vmatrix} \mathbf{k} \]  \hspace{1cm} (A/8)

\[ = (y_a z_b - y_b z_a) \mathbf{i} - (x_a z_b - x_b z_a) \mathbf{j} + (x_a y_b - x_b y_a) \mathbf{k} \]  \hspace{1cm} (A/9)

The two vectors \( \mathbf{a} \wedge \mathbf{b} \) and \( \mathbf{b} \wedge \mathbf{a} \) are pointing away one from the other (fig. A.3):

\[ \mathbf{b} \wedge \mathbf{a} = - (\mathbf{a} \wedge \mathbf{b}) \] \hspace{1cm} (A/10)

Figure A.2 – Two vectors \( \mathbf{a} \) and \( \mathbf{b} \). The vector product \( \mathbf{a} \wedge \mathbf{b} \) has length the product of the lengths \( b_1 \) et \( a \). In the case shown here, the vector \( \mathbf{c} = \mathbf{a} \wedge \mathbf{b} \) is going into through the document plane, going away from the reader.
Figure A.3 – The vectors $\vec{a} \wedge \vec{b}$ and $\vec{b} \wedge \vec{a}$ have the same length but are pointing directions opposite one from the other (the first away from the reader, and the other towards the reader).

If any vector changes direction, the cross product also changes direction (fig. A.4):

$$\vec{a} \wedge -\vec{b} = -(\vec{a} \wedge \vec{b})$$  \hspace{1cm} (A/11)

Figure A.4 – The vector $\vec{a} \wedge -\vec{b}$ is pointing away from the vector $\vec{a} \wedge \vec{b}$. 

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Four operators which apply on vector or scalar fields are important in fluid mechanics: gradient, divergent, Laplacian and curl.

### A3.1 Gradient

The mathematical operator gradient (first introduced as eq. 4/11 p. 79) is written \( \nabla \). It applies on a scalar field and produces a vector field. It is defined as:

\[
\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k \quad (A/12)
\]

\[
\nabla A = \frac{\partial A_x}{\partial x} i + \frac{\partial A_y}{\partial y} j + \frac{\partial A_z}{\partial z} k
\]

\[
\begin{pmatrix}
\frac{\partial A_x}{\partial x} \\
\frac{\partial A_y}{\partial y} \\
\frac{\partial A_z}{\partial z}
\end{pmatrix}
\]

\( (A/13) \)

For example, the gradient of a pressure field is the vector field \( \nabla p \):

\[
\nabla p = \frac{\partial p}{\partial x} i + \frac{\partial p}{\partial y} j + \frac{\partial p}{\partial z} k
\]

\[
\begin{pmatrix}
\frac{\partial p}{\partial x} \\
\frac{\partial p}{\partial y} \\
\frac{\partial p}{\partial z}
\end{pmatrix}
\]

\( (A/14) \)

### A3.2 Divergent

The mathematical operator divergent (first introduced as eq. 5/14 p. 97) is written \( \nabla \cdot \) and is defined as:

\[
\nabla \cdot = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \quad (A/15)
\]

When applied on a vector field, it produces a scalar field:

\[
\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} \hat{i} \cdot \vec{A} + \frac{\partial A_y}{\partial y} \hat{j} \cdot \vec{A} + \frac{\partial A_z}{\partial z} \hat{k} \cdot \vec{A}
\]

\( (A/16) \)

\[
\begin{pmatrix}
\frac{\partial A_x}{\partial x} \\
\frac{\partial A_y}{\partial y} \\
\frac{\partial A_z}{\partial z}
\end{pmatrix}
\]

\( (A/17) \)

When applied on a 2\(^{nd}\) order tensor field, it produces a vector field:

\[
\nabla \cdot \vec{A}_{ij} = \begin{pmatrix}
\frac{\partial A_{xx}}{\partial x} + \frac{\partial A_{yx}}{\partial y} + \frac{\partial A_{zx}}{\partial z} \\
\frac{\partial A_{xy}}{\partial x} + \frac{\partial A_{yy}}{\partial y} + \frac{\partial A_{zy}}{\partial z} \\
\frac{\partial A_{xz}}{\partial x} + \frac{\partial A_{yz}}{\partial y} + \frac{\partial A_{zz}}{\partial z}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\nabla \cdot \vec{A}_{ix} \\
\nabla \cdot \vec{A}_{iy} \\
\nabla \cdot \vec{A}_{iz}
\end{pmatrix}
\]

\( (A/18) \)

For example, the divergent of a velocity field is the scalar field \( \nabla \cdot \vec{V} \):

\[
\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}
\]

\( (A/19) \)
A3.3  Advective

The *advective operator* \( \nabla \cdot \vec{V} \) is defined as follows:

\[
\vec{V} \cdot \vec{\nabla} = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}
\]  

(A/20)

Do not confuse the advective operator with the divergent of velocity, \( \nabla \cdot \vec{V} \) (see Appendix A3.2 above, including eq. A/19), which is a scalar field.

The advective operator can be applied to a scalar field \( A \):

\[
(\vec{V} \cdot \vec{\nabla})A = u \frac{\partial A}{\partial x} + v \frac{\partial A}{\partial y} + w \frac{\partial A}{\partial z}
\]  

(A/21)

It can also be applied to a vector field \( \vec{A} \):

\[
(\vec{V} \cdot \vec{\nabla})\vec{A} = u \frac{\partial \vec{A}}{\partial x} + v \frac{\partial \vec{A}}{\partial y} + w \frac{\partial \vec{A}}{\partial z}
\]  

(A/22)

\[
\begin{pmatrix}
\frac{\partial \vec{A}_x}{\partial x} + \frac{\partial \vec{A}_y}{\partial y} + \frac{\partial \vec{A}_z}{\partial z} \\
\frac{\partial \vec{A}_y}{\partial x} + \frac{\partial \vec{A}_z}{\partial y} + \frac{\partial \vec{A}_x}{\partial z} \\
\frac{\partial \vec{A}_z}{\partial x} + \frac{\partial \vec{A}_x}{\partial y} + \frac{\partial \vec{A}_y}{\partial z}
\end{pmatrix}
\]  

(A/23)

A3.4  Laplacian

The mathematical operator *Laplacian* \( \nabla^2 \) (first introduced as eq. 6/38 p. 126) is written \( \nabla^2 \) and defined as:

\[
\nabla^2 = \nabla \cdot \nabla
\]  

(A/24)

When applied to a scalar field, it is equal to the divergent of the gradient of the field, and produces a scalar field:

\[
\nabla^2 A = \nabla \cdot \nabla A = \frac{\partial^2 A}{(\partial x)^2} + \frac{\partial^2 A}{(\partial y)^2} + \frac{\partial^2 A}{(\partial z)^2}
\]  

(A/25)

(A/26)

When applied to a vector field, the general expression uses the curl operator (we never use this expression in this course), and produces a vector field:

\[
\nabla^2 \vec{A} = (\nabla \cdot \nabla) \vec{A} - \nabla \times (\nabla \times \vec{A})
\]  

(A/27)

In Cartesian coordinates, this simplifies as:

\[
\nabla^2 \vec{A} = \begin{pmatrix}
\nabla^2 A_x \\
\nabla^2 A_y \\
\nabla^2 A_z
\end{pmatrix} = \begin{pmatrix}
\nabla \cdot \nabla A_x \\
\nabla \cdot \nabla A_y \\
\nabla \cdot \nabla A_z
\end{pmatrix}
\]  

(A/28)

\[
\begin{pmatrix}
\frac{\partial^2 A_x}{(\partial x)^2} + \frac{\partial^2 A_x}{(\partial y)^2} + \frac{\partial^2 A_x}{(\partial z)^2} \\
\frac{\partial^2 A_y}{(\partial x)^2} + \frac{\partial^2 A_y}{(\partial y)^2} + \frac{\partial^2 A_y}{(\partial z)^2} \\
\frac{\partial^2 A_z}{(\partial x)^2} + \frac{\partial^2 A_z}{(\partial y)^2} + \frac{\partial^2 A_z}{(\partial z)^2}
\end{pmatrix}
\]  

(A/29)
For example, the Laplacian of a velocity field is the vector field $\nabla^2 \vec{V}$:

$$\nabla^2 \vec{V} = \begin{pmatrix} \vec{\nabla}^2 u \\ \vec{\nabla}^2 v \\ \vec{\nabla}^2 w \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{pmatrix}$$

(A/30)

### A3.5 Curl

The mathematical operator $\text{curl}$ (sometimes named *rotational*) is written $\nabla \times$. It applies to a vector field and produces a vector field. It is defined as:

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{i} + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{j} + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{k}$$

(A/31)

For example, the curl of velocity is the vector field $\nabla \times \vec{V}$:

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{i} + \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \hat{j} + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k}$$

(A/33)
A4 Derivations of the Bernoulli equation

A4.1 The Bernoulli equation from the energy equation

This is covered in section 2.6 p. 43.

A4.2 The Bernoulli equation from the integral momentum equation

We begin with the integral linear momentum equation (eq. 3/9 p. 58):

\[ \vec{F}_{\text{net}} = \frac{d}{dt} \iint_{CV} \rho \vec{V} \, dV + \int_{CS} \rho \vec{V} \cdot (\vec{V}_{\text{rel}} - \vec{n}) \, dA \]

When considering a fixed, infinitely short control volume along a known streamline \( s \) of the flow, this equation becomes:

\[ d\vec{F}_{\text{pressure}} + d\vec{F}_{\text{shear}} + d\vec{F}_{\text{gravity}} = \frac{d}{dt} \iint_{CV} \rho \vec{V} \, dV + \rho VA \, d\vec{V} \]

along a streamline, where the velocity \( \vec{V} \) is aligned (by definition) with the streamline.

Now, adding the restrictions of steady flow (\( \frac{d}{dt} = 0 \)) and no friction (\( d\vec{F}_{\text{shear}} = \vec{0} \)), we already obtain:

\[ d\vec{F}_{\text{pressure}} + d\vec{F}_{\text{gravity}} = \rho VA \, d\vec{V} \]

The projection of the net force due to gravity \( d\vec{F}_{\text{gravity}} \) on the streamline segment \( ds \) has norm \( d\vec{F}_{\text{gravity}} \cdot d\vec{s} = -g \rho A \, dz \), while the net force due to pressure is aligned with the streamline and has norm \( d\vec{F}_{\text{pressure}} \parallel s = -A \, dp \).

Along this streamline, we thus have the following scalar equation, which we integrate from points 1 to 2:

\[ -A \, dp - \rho g A \, dz = \rho VA \, dV \]
\[ -\frac{1}{\rho} \, dp - g \, dz = V \, dV \]
\[ -\int_1^2 \frac{1}{\rho} \, dp - \int_1^2 g \, dz = \int_1^2 V \, dV \]

The last obstacle is removed when we consider flows without heat or work transfer, where, therefore, the density \( \rho \) is constant. In this way, we arrive to equation 2/20 p. 44 again:

\[ \frac{p_1}{\rho} + \frac{1}{2} V_1^2 + gz_1 = \frac{p_2}{\rho} + \frac{1}{2} V_2^2 + gz_2 \]
\[ \left( p + \frac{1}{2} \rho V^2 + \rho g z \right)_1 = \left( p + \frac{1}{2} \rho V^2 + \rho g z \right)_2 \]

A4.3 The Bernoulli equation from the Navier-Stokes equation

We start by following a particle along its path in an arbitrary flow, as displayed in fig. A.5. The particle path is known (condition 5 in §2.6 p. 43), but its speed \( V \) is not.
We are now going to project every component of the Navier-Stokes equation (eq. 6/42 p. 127) onto an infinitesimal portion of trajectory $d\vec{s}$. Once all terms have been projected, the Navier-Stokes equation becomes a scalar equation:

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho (\vec{V} \cdot \vec{V}) \vec{V} = \rho \vec{g} - \vec{V} p + \mu \vec{V} \cdot \vec{V}$$

Because the velocity vector $\vec{V}$ of the particle, by definition, is always aligned with the path, its projection is always equal to its norm: $\vec{V} \cdot d\vec{s} = V \, ds$. Also, the downward gravity $g$ and the upward altitude $z$ have opposite signs, so that $\vec{g} \cdot d\vec{s} = -g \, dz$; we thus obtain:

$$\rho \frac{\partial \vec{V}}{\partial t} \, ds + \rho \frac{\partial \vec{V}}{\partial s} V \, ds = -\rho g \, dz - \frac{dp}{ds} \, ds + \mu \vec{V} \cdot \vec{V} \cdot d\vec{s}$$

When we restrict ourselves to steady flow (condition 1 in §2.6), the first left-hand term vanishes. Neglecting losses to friction (condition 4) alleviates us from the last right-hand term, and we obtain:

$$\rho \frac{dV}{ds} \, ds = -\rho g \, dz - \frac{dp}{ds} \, ds$$

This equation can then be integrated from point 1 to point 2 along the pathline:

$$\rho \int_1^2 V \, dV = -\int_1^2 \rho g \, dz - \int_1^2 dp$$

When no work or heat transfer occurs (condition 3) and the flow remains incompressible (condition 2), the density $\rho$ remains constant, so that we indeed have returned to eq. 2/20 p. 44:

$$\Delta \left( \frac{1}{2} V^2 \right) + g \Delta z + \frac{1}{\rho} \Delta p = 0 \quad \text{(A/34)}$$

$$\left( p + \frac{1}{2} \rho V^2 + \rho g z \right)_1 = \left( p + \frac{1}{2} \rho V^2 + \rho g z \right)_2 \quad \text{(A/35)}$$
Thus, we can see that if we follow a particle along its path, in a steady, incompressible, frictionless flow with no heat or work transfer, its change in kinetic energy is due only to the result of gravity and pressure, in accordance with the Navier-Stokes equation.
Instead of the mathematical approach covered in §8.2.2 p. 164, the concept of flow parameter can be approached by comparing forces in fluid flows.

Fundamentally, understanding the movement of fluids requires applying Newton’s second law of motion: the sum of forces which act upon a fluid particle is equal to its mass times its acceleration. We have done this in an aggregated manner with integral analysis (in chapter 3, eq. 3/9 p. 58), and then in a precise and all-encompassing way with differential analysis (in chapter 6, eq. 6/42 p. 127). With the latter method, we obtain complex mathematics suitable for numerical implementation, but it remains difficult to obtain rapidly a quantitative measure for what is happening in any given flow.

In order to obtain this, an engineer or scientist can use force ratios. This involves comparing the magnitude of a type of force (pressure, viscous, gravity) either with another type of force, or with the mass-times-acceleration which a fluid particle is subjected to as it travels. We are not interested in the absolute value of the resulting ratios, but rather, in having a measure of the parameters that influence them, and being able to compare them across experiments.

A5.1 Acceleration vs. viscous forces: the Reynolds number

The net sum of forces acting on a particle is equal to its mass times its acceleration. If a representative length for the particle is \( L \), the particle mass grows proportionally to the product of its density \( \rho \) and its volume \( L^3 \). Meanwhile, its acceleration relates how much its velocity \( \vec{v} \) will change over a time interval \( \Delta t \): it may be expressed as a ratio \( \Delta V/\Delta t \). In turn, the time interval \( \Delta t \) may be expressed as the representative length \( L \) divided by the velocity \( \vec{v} \), so that the acceleration may be represented as proportional to the ratio \( \frac{\vec{v} \Delta V}{L} \).

\[
|\text{net force}| = |\text{mass} \times \text{acceleration}| \sim \rho L^3 \frac{V \Delta V}{L} \\
|\vec{F}_{\text{net}}| \sim \rho L^2 V \Delta V
\]

We now observe the viscous force acting on a particle: it is proportional to the shear effort and a representative acting surface \( L^2 \). The shear can be modeled as proportional to the viscosity \( \mu \) and the rate of strain, which will grow proportionally to \( \Delta V/L \). We thus obtain a crude measure for the magnitude of the shear force:

\[
|\text{viscous force}| \sim \mu \frac{\Delta V}{L} L^2 \\
|\vec{F}_{\text{viscous}}| \sim \mu \Delta V L
\]

The magnitude of the viscous force can now be compared to the net force:

\[
\frac{|\text{net force}|}{|\text{viscous force}|} \sim \frac{\rho L^2 V \Delta V \mu}{|\mu \Delta V L|} = \frac{\mu V L}{\mu} = \text{[Re]} \quad (A/36)
\]
and we recognize the ratio as the Reynolds number (8/12 p. 166). We thus see that the Reynolds number can be interpreted as the inverse of the influence of viscosity. The larger \( Re \) is, and the smaller the influence of the viscous forces will be on the trajectory of fluid particles.

A5.2 Acceleration vs. gravity force: the Froude number

The weight of a fluid particle is equal to its mass, which grows with \( \rho L^3 \), multiplied by gravity \( g \):

\[
|\text{weight force}| = |\vec{F}_w| \sim \rho L^3 g
\]

The magnitude of this force can now be compared to the net force:

\[
\frac{|\text{net force}|}{|\text{weight force}|} \sim \frac{\rho L^2 V^2}{\rho L^3 g} = \frac{V^2}{Lg} = [Fr]^2 \quad (A/37)
\]

and here we recognize the square of the Froude number (8/11 p. 166). We thus see that the Froude number can be interpreted as the inverse of the influence of weight on the flow. The larger \([Fr]\) is, and the smaller the influence of gravity will be on the trajectory of fluid particles.

A5.3 Acceleration vs. elastic forces: the Mach number

In some flows called compressible flows the fluid can perform work on itself, and the fluid particles then store and retrieve energy in the form of changes in their own volume. In such cases, fluid particles are subject to an elastic force in addition to the other forces. We can model the pressure resulting from this force as proportional to the bulk modulus of elasticity \( K \) of the fluid (formally defined as \( K = \rho \partial p/\partial \rho \)); the elastic force can therefore be modeled as proportional to \( KL^2 \):

\[
|\text{elasticity force}| = |\vec{F}_{\text{elastic}}| \sim KL^2
\]

The magnitude of this force can now be compared to the net force:

\[
\frac{|\text{net force}|}{|\text{elasticity force}|} \sim \frac{\rho L^2 V^2}{KL^2} = \frac{\rho V^2}{K}
\]

This ratio is known as the Cauchy number; it is not immediately useful because the value of \( K \) in a given fluid varies considerably not only according to temperature, but also according to the type of compression undergone by the fluid: for example, it grows strongly during brutal compressions.

During isentropic compressions and expansions (isentropic meaning that the process is fully reversible, i.e. without losses to friction, and adiabatic, i.e. without heat transfer), it can be shown that the bulk modulus of elasticity is proportional to the square of the speed of sound \( c \):

\[
K_{\text{reversible}} = c^2 \rho \quad (A/38)
\]

The Cauchy number calibrated for isentropic evolutions is then

\[
\frac{|\text{net force}|}{|\text{elasticity force}|_{\text{reversible}}} \sim \frac{\rho V^2}{K} = \frac{V^2}{c^2} = [Ma]^2 \quad (A/39)
\]
and here we recognize the square of the Mach number (1/10 p. 18). We thus see that the Mach number can be interpreted as the influence of elasticity on the flow. The larger $|\text{Ma}|$ is, and the smaller the influence of elastic forces will be on the trajectory of fluid particles.

### A5.4 Other force ratios

The same method can be applied to reach the definitions for the Strouhal and Euler numbers given in §8.2 p. 163. Other numbers can also be used which relate forces that we have ignored in our study of fluid mechanics. For example, the relative importance of surface tension forces or of electromagnetic forces are quantified using similarly-constructed flow parameters.

In some applications featuring rotative motion, such as flows in centrifugal pumps or planetary-scale atmospheric weather, it may be convenient to apply Newton’s second law in a rotating reference frame. This results in the appearance of new reference-frame forces, such as the Coriolis or centrifugal forces; their influence can then be studied using additional flow parameters.

In none of those cases can flow parameters give enough information to predict solutions. They do, however, provide quantitative data to indicate which forces are relevant in which places: this not only helps us understand the mechanisms at work, but also distinguish the negligible from the influential, a key characteristic of efficient scientific and engineering work.
The final examination for this course in the summer semester 2020 will take place on September 21, 2020 from 12:00 to 14:00 in G26-H1. The most important information is as follows:

- The exam counts 50% towards your final mark. See the syllabus page 7 for details.
- You must register through the university’s LSF system to attend the exam, several weeks ahead of time.
- The exam lasts two full hours (120 minutes).
- It is closed-book (no documents are allowed).

The exam has six problems. Among those, Problem 1 is mandatory and counts 10 pts. You must choose 3 problems among the remaining five (30 pts each). All your answers will be graded, but only the best 3 problems you attempted will count towards your grade.

Exam rules are as follows:

- A formula sheet is provided. It is the sum of the formula sheets present at the start of every problem sheet in the lecture notes. It includes the Moody diagram and the viscosity diagram used in the problem sheets.
- Calculators are allowed, including programmable calculators, but excluding communicating devices.
- No documents are allowed, but language dictionaries are fine.
- Communicating devices, pencil cases, and bags will have to be left on the sides of the examination room.
- You may leave whenever you wish, except in the last 10 minutes for logistics reasons.
The complete list of examinable problems is as follows:

- 2.2 2.3 2.4 2.5 2.6
- 3.2 3.3 3.4 3.5 3.6 3.7 3.8 3.9
- 4.4 4.5 4.6\(^1\) 4.7
- 5.2 5.3 5.4 5.5 5.6
- 6.2 6.3 6.4 6.5 6.6 6.7 6.9
- 7.3 7.4 7.5 7.6 7.7 7.8
- 8.2 8.4 8.5
- Chapter 9 is not examinable this semester
- 10.2 10.3 10.4 10.5 10.6
- Chapter 11 is not examinable this semester

The first, mandatory problem is exercise 6.2 p.133. The five other problems are extracted from the list above, and modified slightly. Typically, the input data is changed, as well as the problem geometry. The method for solving the problems remains the same.

Examinations from previous years, and their full solution, will be released in July on the course website (https://fluidmech.ninja/). Since the course content has changed over time, you might find a few differences:

- Problems involving calculating compressible air flow using tables are no longer examinable;
- A problem involving a ball fountain (“Kugel fountain”) is no longer examinable;
- Viscosity values were read in a different diagram, and may not match values read in the 2020 viscosity diagram.

You are welcome (and in fact encouraged!) to ask me questions of all sorts about the exam. You may contact me as described in the introduction, page 9.

I wish you to have productive and joyful revisions!

Olivier
July 2020

\(^1\)In exercise 4.6, only the calculation of the vertical force \(F_{\text{top}}\) is examinable.
The following pages present the final examination for this course in July 2019, and its full solution.
(1) \[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \]

The speed of sound is modeled as:

\[ c = \sqrt{\frac{K^2}{\rho_0}} \]

The force coefficients and power coefficients are defined as:

\[ \frac{\partial F}{\partial \rho} = \frac{1}{r} \quad \text{and} \quad \frac{\partial P}{\partial \rho} = \frac{1}{r} \]

(2) \[ A_d + dA_d - \frac{D}{tA} = dA \]

(3) \[ \frac{\partial (bh)}{\partial t} + \nabla \cdot (h \mathbf{u}) = 0 \]

(4) \[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \]

(5) \[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \]

The speed of sound in air is modeled as:

\[ c = \sqrt{\frac{1}{\rho} \times \frac{d \rho}{dt}} \]

Air behaves as a perfect gas:

\[ p = \rho R T \]

Non-dimensional incompressible Navier-Stokes equation:

\[ \frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \mathbf{u}) = -\nabla p + \nu \nabla^2 \mathbf{u} \]

Balance of momentum in a considered volume with steady flow:

\[ \int \mathbf{F} dV = \int \mathbf{F} dV \]

Balance of energy in a considered volume with steady flow:

\[ \int \mathbf{Q} dV = \int \mathbf{Q} dV \]

Balance of mass in a considered volume with steady flow:

\[ \int \rho dV = \int \rho dV \]

Balance of net energy in a considered volume with steady flow:

\[ \int \mathbf{W} dV = \int \mathbf{W} dV \]

Balance of net power in a considered volume with steady flow:

\[ \int \mathbf{W} dV = \int \mathbf{W} dV \]

Fluid Mechanics examination — July 11, 2019
In boundary layer flow, we assume that transition occurs at \( Re_x \gtrsim 5 \times 10^5 \).

The wall shear coefficient \( c/u_1 \), a function of distance \( x \), is defined based on the free-stream flow velocity:

\[
c/u_1 = \frac{c}{u_1} \left( \frac{x}{\rho u_1^2} \right)^{1/2}
\]

Exact solutions to the laminar boundary layer above a smooth surface yield the following:

\[
\begin{align*}
\frac{c}{u_1} & = \frac{1.7}{\sqrt[4]{Re}} \\
\frac{c}{u_1} & = \frac{x}{\rho u_1^2}
\end{align*}
\]

Figure 1 – Viscosity of various fluids at a pressure of 1 bar (in practice viscosity is almost independent of pressure).

Figure © White 2008

Figure 2 – A Moody diagram, which presents values for wall shear stress \( c/u_1 \) measured experimentally as a function of the diameter-based Reynolds number \( Re_D \), for different relative roughness values.

Diagram CC/hyphen.sc/b.sc/y.sc/hyphen.sc/s.sc/a.sc S Beck and R Collins, University of Sheffield
1 Governing equation

1.1. [5pts] Write out equation (8), the Navier-Stokes equation for incompressible flow, in its fully-developed form in three Cartesian coordinates.

1.2. [5pts] Write out equation (7), the continuity equation for incompressible flow, in its fully-developed form in three Cartesian coordinates.

2 Observation window in a water tank

A water tank used in a laboratory is filled with stationary water (Fig. 3). A window is installed on one of the walls of the canal to enable observation. The window is hinged with an angle $\theta = 70^\circ$ relative to horizontal. The window has a height of 1.5 m and a width of 3.5 m. The walls of the tank are inclined with an angle $\alpha$ relative to horizontal.

Figure 3 – A window installed on the wall of a water tank.

2.1. [15pts] What is the magnitude of the net force applying on the tank window?

2.2. [10pts] At what distance away from the hinge does this force apply?

2.3. [5pts] How will the distance calculated above change as water is added to the water level increases?

Water is added to the tank, so that the water level increases.

2.3. [5pts] How will the distance calculated above change as water is added? (brieWy justify your answer, e.g. in 30 words or less)

5 Solve problem 1, and three other problems among problems 2 to 6.

The following marking guidelines will be used:

• Answers to questions starting with "show that" should be fully-developed and continuous;
• In all other questions, the correct result with the correct unit is enough to obtain full points;
• Illegible or ambiguous answers are always discarded.

6
A pipe leads water from one reservoir to a turbine, which discharges into another reservoir, as shown in Figure 4.

Figure 4 – Layout of the water pipe. For clarity, in this figure, the vertical scale is greatly exaggerated. In the vertical scale, the diameter of the pipe is also greatly exaggerated.

The pipe is made of coarse concrete (roughness \( \varepsilon = 0.25 \text{mm} \)) and carries 800Ls\(^{-1}\) of water at 20\(\degree\)C. It has a diameter of 1.1m and features four elbow bends with sharp angles, each inducing a loss coefficient \( K_L = 0.75 \).

3.1. \([10\text{pts}]\) Represent qualitatively (i.e. without numerical data) the pressure distribution along the length of the pipe, both when the turbine is shut down (without any flow), and when it is operating.

3.2. \([15\text{pts}]\) What is the hydraulic power available to the turbine?

3.3. \([5\text{pts}]\) When the water stops flowing, what will be the height of the water level in the source tank on the left?

4 Boundary layer on a flat plate
A thin and smooth plate with width \( W = 0.6 \text{m} \) and length \( L = 2 \text{m} \) is placed with a zero angle of attack in atmospheric air flowing incoming at 21ms\(^{-1}\), as shown in Figure 5.

We would like to study the shear stress exerted by the flow over the top surface of the plate.

![Diagram of the plate in the wind tunnel.](image)

Figure 5 – A thin plate positioned parallel to an incoming uniform flow. (credit: sc/0/0/sc.0/c.sc).

4.1. \([5\text{pts}]\) At what distance \( x_{tr.} \) along the plate, approximately, will the boundary layer transit and become turbulent?

4.2. \([10\text{pts}]\) Starting from equation (21), which quantifies the friction factor \( c/u_1\sqrt{\mu}\) in a laminar boundary layer, show that the shear force \( F_{laminar} \) exerted in the laminar section of the boundary layer is:

\[
F_{laminar} = 0.664 \sqrt{\frac{\mu}{\rho}} \frac{W x_{1/2}}{Re_x} (22)
\]

4.3. \([5\text{pts}]\) What is the shear force exerted on the top surface of the plate by the laminar section of the boundary layer?

4.4. \([5\text{pts}]\) What is the shear force exerted on the top surface of the plate by the turbulent section of the boundary layer?

4.5. \([5\text{pts}]\) Would the boundary layer become thicker if the velocity was increased? (brieﬂy justify your answer, e.g. in 30 words or less).
Velocity measurements in a tunnel

A group of students proceeds with speed measurements in a water tunnel. The objective is to measure the drag applying on an object with constant cross-section, positioned across the tunnel test section (Fig. 6).

![Figure 6](image)

**Figure 6** – An object with constant cross-section positioned across a water tunnel. The object spans completely across the tunnel (in the \( \mathbf{z} \)-direction). The horizontal velocity distributions upstream and downstream of the profile are also shown.

Upstream of the object, the water flow velocity is uniform (\( u_1 = 3 \text{ m/s} \)).

Downstream of the object, horizontal velocity measurements are made every 5 cm across the flow:

<table>
<thead>
<tr>
<th>Vertical Position (cm)</th>
<th>Horizontal Speed (m/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3.2</td>
</tr>
<tr>
<td>5</td>
<td>3.3</td>
</tr>
<tr>
<td>10</td>
<td>3.4</td>
</tr>
<tr>
<td>15</td>
<td>3.5</td>
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<td>20</td>
<td>3.6</td>
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<td>25</td>
<td>3.7</td>
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<td>30</td>
<td>3.8</td>
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<td>35</td>
<td>3.9</td>
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<td>4.0</td>
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<tr>
<td>45</td>
<td>4.1</td>
</tr>
<tr>
<td>50</td>
<td>4.2</td>
</tr>
<tr>
<td>55</td>
<td>4.3</td>
</tr>
<tr>
<td>60</td>
<td>4.4</td>
</tr>
</tbody>
</table>

The width of the profile (perpendicular to the flow, in the \( \mathbf{z} \)-direction) is 70 cm. The water has uniform temperature and density (20 °C, 999 kg/m\(^3\)) and the pressure is uniform.

6 Lift and drag on a rotating football

A group of fluid dynamicists investigates the air flow around a football. In particular, they are interested in the forces applying on the ball when it has been kicked and is flying through the air. The football has a diameter of 22 cm, a weight of 430 g; it is traveling at 70 km/h.

In order to observe the flow, they install a steel sphere in a wind tunnel (Fig. 7). The sphere has a diameter of 1 cm. Drag force measurements are carried out in the tunnel.

![Figure 7](image)

**Figure 7** – A steel sphere positioned in a wind tunnel. Force measurements are carried out on the ball.

6.1 [20 pts] What is the drag force applying on the profile?

6.2 [10 pts] If the ball were replaced with a ball with higher viscosity, how would you expect the drag force to change?

The width of the profile is also measured in a tunnel. The objective is to measure the drag applying on an object with constant cross-section, positioned across the tunnel.
6.1. [5 pts] What is the wind tunnel speed required, so that the flow around the real football is reproduced around the sphere in the tunnel?

6.2. [5 pts] With the speed calculated above, by which factor should the drag force measured in the wind tunnel be multiplied, in order to obtain the drag force on the real football?

The fluid dynamicists now investigate the effect of spin on the ball. When the football is rotated along a horizontal axis during travel, a lift force exerts laterally on the ball, curving its trajectory. This is represented from above in Figure 8.

In order to quantify this effect, the wind tunnel sphere is rotated in the wind tunnel, and measurements are carried out; the results are plotted in Figure 9.

6.3. [10 pts] How many rotations per second are required in order to generate a lift force of 3 N on the real football when it travels at a speed of 3.1 m/s?

6.4. [5 pts] When is the corresponding drag force zero?

6.5. [5 pts] What is the wind tunnel speed required, so that the flow around the real football is reproduced around the sphere in the tunnel?

6.6. [5 pts] With the speed calculated above, by which factor should the drag force measured in the wind tunnel be multiplied, in order to obtain the drag force on the real football?
1 Governing equation

1.1 N-S equation question

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{\rho} \left( \frac{\partial (\mu \nabla^2 u)}{\partial x} + \frac{\partial (\mu \nabla^2 v)}{\partial y} + \frac{\partial (\mu \nabla^2 w)}{\partial z} \right) \]  

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{1}{\rho} \left( \frac{\partial (\mu \nabla^2 v)}{\partial x} + \frac{\partial (\mu \nabla^2 u)}{\partial y} + \frac{\partial (\mu \nabla^2 w)}{\partial z} \right) \]  

\[ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{1}{\rho} \left( \frac{\partial (\mu \nabla^2 w)}{\partial x} + \frac{\partial (\mu \nabla^2 u)}{\partial y} + \frac{\partial (\mu \nabla^2 v)}{\partial z} \right) \]  

1.2 Continuity question

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]  

2 Observation window in a water tank

2.1 Net force

We define coordinates \( r \) and \( \theta \) and the length \( L \), as shown in the figure below.

The force on a small section of door with length \( d \theta \) and width \( W \), \( dF \), on the complete door, the force applying due to the net pressure \( p_{\text{net}} \) of water and air is

\[ F_{\text{net}} = \int_{0}^{L} \int_{0}^{2\pi} \rho g z \, W \, dr \, d\theta \]  

\[ F_{\text{net}} = \int_{0}^{L} \int_{0}^{2\pi} \rho g z \, W \, dr \, d\theta \]  

\[ F_{\text{net}} = \int_{0}^{L} \int_{0}^{2\pi} \rho g Z \, W \, dr \, d\theta \]  

\[ F_{\text{net}} = \int_{0}^{L} \int_{0}^{2\pi} \rho g Z \, W \, dr \, d\theta \]  

A coordinate transform is needed to solve the integral, expressing \( r \) as a function of \( \theta \).
Inserting eq. 9 into eq. 8, we continue with:

\[ \sum_{r=0}^{\text{max}} r \Delta g(\text{min} + x) \text{d}x \]

We water, using the same notation as above:

The distance away from the hinge is obtained by dividing the moment by the force:

\[ \sum_{r=0}^{\text{max}} r \Delta g(\text{min} + x) \text{d}x \]

\[ \sum_{r=0}^{\text{max}} r \Delta g(\text{min} + x) \text{d}x \]

We can calculate the moment exerting about the hinge due to the net pressure of air and increase together with \( l \) (eqs. 14 & 22), so it is not immediately apparent how changes in \( l \) and \( x \) relate to each other. Below, both \( l \) and \( x \) increase together with \( l \). The changes in \( l \) lead to the change in \( x \) in the equations.

### 2.3 Distance from Hinge

\[ \sum_{r=0}^{\text{max}} r \Delta g(\text{min} + x) \text{d}x \]

\[ \sum_{r=0}^{\text{max}} r \Delta g(\text{min} + x) \text{d}x \]

\[ \sum_{r=0}^{\text{max}} r \Delta g(\text{min} + x) \text{d}x \]
3 Piping leading to a turbine

3.1 Pressure distribution

3.2 Turbine power

We want to calculate three pressure drops:

1. Pressure drop due to wall friction losses along the pipe,

\[ p_{u1D453} : \] The average velocity in the pipe is

\[ u_{av} = \frac{\mu}{\rho} \gamma \delta \] (32)

\[ = \frac{\mu}{\rho} \gamma \delta D^2 \] (33)

\[ = 0,8 \times 1,12 \] (34)

\[ u_{av} = 0,842 \text{ms}^{-1} \] (35)

The Reynolds number is

\[ [Re]D = \frac{\mu u_{av}}{\nu} \] (36)

\[ = 10^3 \times 0,842 \times 1,1 \times 10^{-5} \] (37)

\[ [Re]D = 9,251 \times 10^5 \] (38)

The relative roughness is

\[ \delta / \delta D = 0,25 \] (39)

\[ \delta / \delta D = 2,27 \times 10^{-4} \] (40)

With those values, the Moody diagram reads:

\[ \frac{\mu}{\rho} \gamma \delta \] (41)

Finally, the wall friction losses along the pipe are calculated as

\[ p_{u1D453} = -0,0158 \] (42)

\[ p_{u1D453} = -2 \times 10^4 \text{Pa} \] (44)

2. Pressure drop due to losses in the four bends,

\[ p_{bends} = -4 \times \frac{K}{L} \] (45)

\[ = -4 \times 0,75 \times \frac{1}{2} \times 10^3 \times 0,842 \] (46)

\[ p_{bends} = -1 \times 10^3 \text{Pa} \] (47)

3. Pressure drop due to hydrostatic pressure change across the turbine,

\[ p_{h} : \] The average velocity along the pipe

\[ \frac{\mu}{\rho} \gamma \delta \] (48)

\[ = 10^3 \times 9,81 \times 4 - (25 + 51/ \] (49)

\[ p_{h} = -7 \times 10^5 \text{Pa} \] (50)

Finally, the wall friction losses along the pipe are calculated as:

\[ p_{u1D453} = -0,1 \text{ms}^{-1} \] (51)

\[ p_{u1D453} = -0,09 \times 10^3 \text{Pa} \] (52)

\[ p_{u1D453} = -0,07 \times 10^3 \text{Pa} \] (53)

\[ p_{u1D453} = -0,06 \times 10^3 \text{Pa} \] (54)

\[ p_{u1D453} = -0,05 \times 10^3 \text{Pa} \] (55)
Finally, the turbine hydraulic power is obtained as:

$$W_{turbine} = \frac{\Delta p_{turbine}}{51}$$

$$= \frac{p_h - p_{bends}}{52}$$

$$= 0,8 \times -7,06 \times 10^5 - (4,03 \times 10^4) - (1,06 \times 10^{-3})$$

$$= -5,479 \times 10^5 W$$

$$= -547,9 kW$$

3.3 Residual water height
The water will flow until air is entrained ("sucked") into the pipe inlet. At this point, the residual water height in the left tank will be 8m.

4 Boundary layer on a Wat plate
4.1 Transition point
The transition point occurs at $Re_x \approx 5 \times 10^5$. Solving for $x_{tr}$, we have:

$$Re_x = \frac{u_{inf} x_{tr}}{v}$$

$$x_{tr} = \frac{Re_x}{\frac{u_{inf}}{v}}$$

$$x_{tr} = 0,29 m$$

4.2 Shear in laminar section
We start with the given equation and implement the definition (15) of the formula sheet, as well as the definition of the distance-based Reynolds number $Re_x$.

The shear force is the integral of the shear with respect to area:

$$\int \frac{dF_{wall, laminar}}{dx} = \int \frac{d\tau}{dx}$$

We split the total area covered by the laminar boundary layer into strips of width $W$ and length $dx$.
4.3 Value of shear in the laminar section

We simply insert values into eq. 70:

\[ \text{shear, laminar} = 0, \frac{664}{\sqrt{13 \times 10^{-5}} \times 0,292} \times (1,5 \cdot 10^{-5}) \times 0,6 \times 12 \times 0,01575 \times 0,0887 \text{N} \] (78)

4.4 Shear in turbulent section

The process is the same in the turbulent part of the layer, with the turbulent section beginning at \( x = x_{\text{tr.}} \) (where the turbulent section begins) and ending at \( x = x_{\text{max}} \) (the trailing edge of the plate):

This is integrated with respect to area, with \( x \) ranging from \( x_{\text{tr.}} \) to \( x_{\text{max}} \):

\[ \text{shear, turbulent} = \int_{x_{\text{tr.}}}^{x_{\text{max}}} \frac{\partial z}{\partial x} \, dx = 0, \frac{664}{\sqrt{13 \times 10^{-5}} \times 0,292} \times (1,5 \cdot 10^{-5}) \times 0,6 \times 12 \times 0,01575 \times 0,0887 \text{N} \] (78)

4.5 Thickness

Models for the boundary layer thickness are given in the formula sheet as equations 16 and 18 for one for the laminar section, the other for the turbulent section). In both, the Reynolds number \( \text{Re}_x \) appears in the denominator (the lower part of the fraction). As \( \text{Re}_x \) is increased, \( \text{Re}_x \) will increase too, and consequently, the thickness of the boundary layer will decrease.
We insert the expression for $h_1$ obtained above in this last expression, continuing as:

$$ \Delta \rho \int \gamma d\gamma = \frac{1}{m} F $$

This force is the force exerted on the object on the fluid, and so is pointing in the opposite direction (flow-wise direction):

$$ \Delta \rho \int \gamma d\gamma = -186,45 N $$

5 Velocity measurements in a tunnel

5.1 Drag force

We build a control volume around the object:

We use a mass balance equation which reduces to a scalar equation in the $x$-direction:

$$ \Delta \rho \int \gamma d\gamma - \frac{1}{m} F = \frac{v_y}{m} $$

The drag force is generated using a momentum balance equation which reduces to:

$$ \Delta \rho \int \gamma d\gamma - \frac{1}{m} F = \frac{v_y}{m} $$

We use the mass balance equation to quantify the relation $F$ of the object.

5.1 Drag force

We build a control volume around the object:
5.2 Dependence on viscosity

Viscosity does not appear in equation (90) above. Nevertheless, an increase in viscosity will translate into higher shear, and so it is likely that the object will affect a larger amount of fluid around itself. This will result in a larger velocity deficit, regardless of the actual increase in viscosity. 

6 Lift and drag on a rotating football

6.1 Required wind tunnel speed

The two flows will have identical behavior if the Reynolds numbers are equal. With $Re_1$ denoting the real football, and $Re_2$ denoting the wind tunnel sphere, we have:

$$Re_1 = Re_2 \tag{95}$$

$$C_{L1}/D_1 = C_{L2}/D_2 \tag{96}$$

$$L_1^2 = L_2^2 \tag{97}$$

$$C_{D1}/C_{D2} = \frac{1}{2} \frac{\rho L_1^2}{\rho L_2^2} \tag{98}$$

$$\frac{1}{2} \frac{\rho L_1^2}{\rho L_2^2} = \frac{1}{2} \frac{\rho L_1^2}{\rho L_2^2} \tag{99}$$

$$\frac{1}{2} \frac{\rho L_1^2}{\rho L_2^2} = 14 \text{ km/h} \tag{100}$$

Since the two flows are dynamically similar, the force coefficients are the same. Consider the lift coefficients,

$$C_{L1} = C_{L2} \tag{101}$$

$$\frac{L_1}{D_1} = \frac{L_2}{D_2} \tag{102}$$

$$L_1 = L_2 \tag{103}$$

$$L_1^2 = L_2^2 \tag{104}$$

$$L_1 = L_2 \tag{105}$$

$$L_1 = L_2 \tag{106}$$

So, the forces will be identical on both the wind tunnel model and the real football.
6.3 Rotation speed

The desired lift force is \( L_1 = 3 \text{N} \). This corresponds to a lift coefficient of:

\[
C_{L1} = \frac{L_1}{\frac{1}{2} \rho A D_1^2 u_1^2} = \frac{3}{\frac{1}{2} \times 0.5 \times 1.5 \times 0.5} = 1.51
\]

Inputting this value in Figure 9, one corresponding value of \( u \) is \( 1.51 \text{ m/s} \). This allows us to obtain the rotation speed of \( \omega \), which corresponds to a higher value of \( \theta \) in Figure 9.

6.4 Drag force

The chosen value of \( \frac{1}{2} \rho A D_1^2 u_1^2 \) corresponds to a drag coefficient reading of 0.56 in Figure 9.

\[
C_D = \frac{1}{2} \frac{1}{2} \rho A D_1^2 u_1^2 = \frac{0.56 \times 1.5 \times 0.5}{1.5 \times 0.5} = 1.51
\]

Inputting this value in Figure 9, we can solve for the drag force.

6.5 Doubling of lift force

The force is "easily" doubled by multiplying the speed \( u_1 \) by a factor \( \sqrt{2} \):

\[
u_3 = \sqrt{2} u_1
\]

We obtain the same lift coefficient \( C_{L3} = C_{L1} = 0.352 \). The rotation speed has to be doubled according to the expression 112: we obtain \( \omega_3 = 60.1 \text{ rotations/s} \).

6.3 Rotation speed

The desired lift force is \( L_1 = 3 \text{N} \). This corresponds to a lift coefficient of:

\[
C_{L1} = \frac{L_1}{\frac{1}{2} \rho A D_1^2 u_1^2} = \frac{3}{\frac{1}{2} \times 0.5 \times 1.5 \times 0.5} = 1.51
\]

Inputting this value in Figure 9, one corresponding value of \( u \) is \( 1.51 \text{ m/s} \). This allows us to obtain the rotation speed of \( \omega \), which corresponds to a higher value of \( \theta \) in Figure 9.
These notes are based on textbooks by White [23], Çengel & al.[26], Munson & al.[30], and de Nevers [18].


